Multidimensional Signal Analysis
Part I: Signal Processing

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1 A preview: signal processing

- Fourier transform
- z transform
- linear systems
- parametric spectral density estimation
- nonparametric spectral density estimation
- independent component analysis
- time-frequency decompositions: short-time FT, wavelet transform, Wigner-Ville distribution
- interpolation & sampling
- image processing

2 An extended preview: image processing

A possible description of tasks:

<table>
<thead>
<tr>
<th>input</th>
<th>output</th>
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<tbody>
<tr>
<td>image processing</td>
<td>image</td>
</tr>
<tr>
<td>image analysis</td>
<td>image features, measurements</td>
</tr>
<tr>
<td>scene understanding</td>
<td>image abstract description</td>
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</tbody>
</table>
3 Point operations

act on individual pixel, though information from the histogram is sometimes used.

Examples are contrast-enhancing transformations or thresholding operations.

Original and linear stretch

Original and approximate histogram equalization

Thresholding
4 Linear filtering

The workhorse of image processing.

A picture (left) is convolved with a filter (middle) to give another image (right).

Boundary conditions

What to do with the pixels at the border of an image?

- omit them and leave a blank stripe
- replicate the image; typically (as in the fast Fourier transform):
• Alternatively:

5 Smoothing

is used to reduce the effect of noise or to eliminate confounding information.

A great variety of filters have been developed.

• The Gaussian is among the oldest: its transfer function is also a Gaussian, high spatial frequencies are attenuated with a weight $\propto e^{-\omega^2}$.

• The median is a nonlinear filter that is more robust against outliers.
There are several ways to perform an edge-preserving smoothing. One of the simplest is to use the nonlinear Kuwahara filter.

The mean gray value and variance are computed for each of the four regions surrounding the center pixel. That pixel is then assigned the mean gray value of the region with the lowest variance.

Anisotropic diffusion is another edge-preserving operation. The basic idea is to average parallel to, but not across an edge. The desired behavior is exhibited by a diffusion process with a diffusivity that is non-homogeneous and anisotropic.
In its simplest form, the Wiener filter assumes the image is degraded by additive white noise. The deviations from the mean in a local neighborhood specified by the mask size are multiplied by

\[ \frac{\sigma_n^2 - \sigma_s^2}{\sigma_s^2} \]

where \( \sigma_n^2 \) is the variance of the noise and \( \sigma_s^2 \) the variance in the local neighborhood.
6 Edge detection

can be performed by means of derivative filters. These are sensitive to noise.

The next slide shows derivative images in both directions as well as the gradient magnitude.

Original with added white Gaussian noise
Alternatively, edges can be identified as zero crossings of the curvature (second derivatives).

The estimation of these higher derivatives is even more susceptible to noise, as shown on the next slide: second derivatives in horizontal and vertical direction and Laplacian.

Example: preprocessing of TEM images

The above techniques have been used to register (that is, superimpose) and rectify a number of transmission electron microscopic images of the same sample. This is a prerequisite for further analysis.

7 Morphological image processing

Morphology focuses on the shape of objects. The structuring element is to mathematical morphology what the filter is to signal processing.

The structuring element may have an arbitrary shape, the most typical ones are depicted below.

The dilation operator replaces the central pixel with the maximum value found under the structuring element. The erosion operator uses the minimum value instead.
Example: application in granulometry

Original

After thresholding

dilated

eroded

morph. gradient

Size distribution can be obtained by sieving out larger objects and then measuring their area (using “propagation”, another morphological technique).

The Euclidean distance transform

Within a solid shape, it finds the distance to the closest part of the boundary.

“How far is it to the ocean?”
Applications of the distance transform

Skeletonization

Separation of overlapping particles
(in conjunction with the watershed transform)

The watershed transform
8 Signal processing

8.1 Types of signals
Interpreting signals as a map \(X \rightarrow G\), they can be classified according to

- their domain
  \(X \subseteq \mathbb{Z}\) → discrete-time signals
  \(X \subseteq \mathbb{R}\) → continuous-time signals
- their codomain
  \(G \subseteq \mathbb{Z}\) → quantized signals
  \(G \subseteq \mathbb{R}\) → real signals
  \(G \subseteq \mathbb{C}\) → complex signals

analog signals: \(X, G \subseteq \mathbb{R}\)
digital signals: \(X, G \subseteq \mathbb{Z}\)

Signals can be classified according to their norm:

- finite norm
  energy = \(\int_{-\infty}^{\infty} g^*(x)g(x)dx\) ⇒ energy signals \(\in L^2(\mathbb{R})\)
- infinite norm
  power = \(\lim_{X \rightarrow \infty} \frac{1}{2X} \int_{-X}^{X} g^*(x)g(x)dx\) ⇒ power signals

examples are: periodic functions, step function, infinite-duration noise

8.2 Inner product

- continuous-time signals: \(\langle g, h \rangle = \int_{-\infty}^{\infty} g^*(x)h(x)dx\)
- discrete-time signals: \(\langle g, h \rangle = \sum_{k=-\infty}^{\infty} g^*(k)h(k)\)

8.3 Norm

Squared norm is given by inner product of a signal with itself

\[ ||g||^2 = \langle g, g \rangle \]

8.4 Series expansion

Expand a continuous-time signal \(g(x)\) in basis functions \(\phi_i(x)\) for \(x \in X\).

\[ g(x) = \sum_{i} c_i \phi_i(x) \]

Coefficients are found by solving a system of linear equations:

\[ \int_{X} \phi_i^*(x)g(x)dx = c_1 \int_{X} \phi_i^*(x)\phi_1(x)dx + c_2 \int_{X} \phi_i^*(x)\phi_2(x)dx + \ldots \]

The solution is greatly simplified if the basis functions are orthogonal in the interval \(X\), i.e.

\[ \int_{X} \phi_m^*(x)\phi_n(x)dx = \begin{cases} 0 : m \neq n \\ K_m : m = n \end{cases} \]
In that case,

\[ c_l = \frac{\int_X \phi_l^*(x) g(x) dx}{\int_X \phi_l(x) \phi_l(x) dx} \]

These coefficients minimize the integrated square error (ISE) because

\[ ISE = \int_X \left(g(x) - \sum_l c_l \phi_l(x) \right)^2 dx \]

\[ = \int_X g^*(x) g(x) - 2 \sum_l c_l g^*(x) \phi_l(x) + \sum_l \sum_m c_l c_m \phi_l^*(x) \phi_m(x) dx \]  \hspace{1cm} (1)

and, using orthogonality,

\[ \frac{\partial ISE}{\partial c_n} = \int_X -2g^*(x) \phi_n(x) + 2c_n \phi_n^*(x) \phi_n(x) dx = 0 \]

\[ c_n = \frac{\int_X \phi_n^*(x) g(x) dx}{\int_X \phi_n^*(x) \phi_n(x) dx} \]

as above.

If the series converges, i.e. \( g(x) = \sum_l c_l \phi_l(x) \) then Parseval’s relation

\[ energy = \int_X g^*(x) g(x) dx = \sum_l c_l^2 K_l \]

follows from (1).

There is a multitude of orthogonal series expansions:

- Fourier

- series that orthogonalize the polynomial system 1, \( x, x^2, \ldots \) with different weighting functions
  - Legendre
  - Tchebychev
  - Jacobi
  - Laguerre
  - Hermite

- discrete orthogonal systems
  - Haar
  - Rademacher
  - Walsh
9 Fourier transform

is the most important of all orthogonal expansions. Different definitions exist:

\[ G(\omega) = \int_{-\infty}^{\infty} g(x)e^{-j2\pi\omega x} \, dx \quad \leftrightarrow \quad g(x) = \int_{-\infty}^{\infty} G(\omega)e^{j2\pi\omega x} \, d\omega \]

\[ G(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x)e^{-j\omega x} \, dx \quad \leftrightarrow \quad g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G(\omega)e^{j\omega x} \, d\omega \]

where \( e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i \sin(y)) \).

The transform exists if \( \int |g(x)| \, dx < \infty \), and if \( g(x) \) has bounded variation and a finite number of discontinuities.

9.1 Fourier transform of an even or odd function

Since \( x \in \mathbb{R} \), the transform is

\[ \int (\Re(g(x)) + j\Im(g(x))) (\cos(-2\pi\omega x) + j \sin(-2\pi\omega x)) \, dx \]

If \( g(x) \) is even \( g(x) = g(-x) \), the odd terms disappear when integrated over and the transform \( G(\omega) \) is also even.

Similarly, an odd \( g(x) = -g(-x) \) gives an odd \( G(\omega) \).

\( \Rightarrow \) Even functions and odd functions can be transformed using the cosine and sine transform, respectively.

\[ \int_{-\infty}^{\infty} g_e(x) \cos(-2\pi\omega x) \, dx = 2 \int_{0}^{\infty} g_e(x) \cos(2\pi\omega x) \, dx \]

\[ j \int_{-\infty}^{\infty} g_o(x) \sin(-2\pi\omega x) \, dx = -2j \int_{0}^{\infty} g_o(x) \sin(2\pi\omega x) \, dx \]

Any function \( g(x) \) can be decomposed uniquely into its even and odd parts:

\[ g_e(x) = \frac{1}{2} (g(x) + g(-x)) \]

\[ g_o(x) = \frac{1}{2} (g(x) - g(-x)) \]

so that the Fourier transform of any function \( g(x) = g_e(x) + g_o(x) \) is given by

\[ 2 \int_{0}^{\infty} g_e(x) \cos(2\pi\omega x) \, dx - 2j \int_{0}^{\infty} g_o(x) \sin(2\pi\omega x) \, dx \]

A real function has a Hermitian Fourier transform whose real part is even and whose imaginary part is odd.
9.2 Addition theorem
If \( g(x) \leftrightarrow G(\omega) \) and \( h(x) \leftrightarrow H(\omega) \) then
\[
c_1g(x) + c_2h(x) \leftrightarrow c_1G(\omega) + c_2H(\omega)
\]

9.3 Shift theorem
If \( g(x) \leftrightarrow G(\omega) \) then
\[
g(x - c) \leftrightarrow e^{-j2\pi\omega c}G(\omega)
\]

because
\[
\int_{-\infty}^{\infty} g(x - c)e^{-j2\pi\omega x}dx = \int_{-\infty}^{\infty} g(x)e^{-j2\pi\omega(x-c)}e^{-j2\pi\omega c}dx
\]
\[
= \int_{-\infty}^{\infty} g(x - c)e^{-j2\pi\omega(x-c)}d(x - c)e^{-j2\pi\omega c}
\]
\[
= G(\omega)e^{-j2\pi\omega c}
\]

\( \Rightarrow G(\omega) \) is “twisted” in the complex plane

9.4 Similarity theorem
If \( g(x) \leftrightarrow G(\omega) \) then
\[
g(cx) \leftrightarrow \frac{1}{|c|}G\left(\frac{\omega}{c}\right)
\]

because
\[
\int_{-\infty}^{\infty} g(cx)e^{-j2\pi\omega x}dx = \int_{-\infty}^{\infty} g(x)e^{-j2\pi\left(\frac{\omega}{c}\right)x}dx
\]
\[
= \frac{1}{|c|}\int_{-\infty}^{\infty} g(x)e^{-j2\pi\left(\frac{\omega}{c}\right)x}dx
\]
\[
= \frac{1}{|c|}G\left(\frac{\omega}{c}\right)
\]

9.5 Convolution
Convolution ("Faltung") used to be known under a variety of names, including “smoothing”, “blurring”, “scanning”, “smearing”. It is defined by
\[
(g * h)(x) = \int_{-\infty}^{\infty} g(u)h(x-u)du
\]

and gives the value of \( g \) at the position \( x \), weighted locally with \( h \).

\[
(1,3,2,5,1,1,3,1,1) * (1,1,1) = (1,4,6,10,8,7,5,5,2,1)
\]
9.6 Central limit theorem

9.7 Convolution Theorem

Convolution in space corresponds to multiplication in reciprocal space because

\[ F(g * h) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x')h(x-x')dx'e^{-j2\pi\omega x} \, dx \]

\[ = \int_{-\infty}^{\infty} g(x') \int_{-\infty}^{\infty} h(x-x')e^{-j2\pi\omega x} \, dx \, dx' \]

\[ = \int_{-\infty}^{\infty} g(x')e^{-j2\pi\omega x'} \, dx' H(\omega) \]

\[ = G(\omega)H(\omega) = F(g)F(h) \]

9.8 Rayleigh’s theorem

corresponds to Parseval’s theorem which holds for Fourier series

\[ \text{energy} = \int_{-\infty}^{\infty} |g(x)|^2 \, dx = \int_{-\infty}^{\infty} |G(\omega)|^2 \, d\omega \]

because

\[ \int_{-\infty}^{\infty} g^*(x)g(x) \, dx = \int_{-\infty}^{\infty} g^*(x)g(x)e^{-j2\pi\omega' x} \, dx \bigg|_{\omega'=0} \]

\[ = G^*(-\omega')*G(\omega') \bigg|_{\omega'=0} \]

\[ = \int_{-\infty}^{\infty} G^*(\omega-\omega')G(\omega) \, d\omega \bigg|_{\omega'=0} \]

\[ = \int_{-\infty}^{\infty} G^*(\omega)G(\omega) \, d\omega \]

9.9 Wiener-Khintchine theorem

relates the power spectral density to the (auto-) correlation function

\[ S_{gg}(\omega) = \int_{-\infty}^{\infty} \text{Corr} (g(x), g(x+x')) e^{-j2\pi\omega x'} \, dx' \]

\[ = F(\text{Corr} (g(x), g(x+x')))) \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g^*(x)g(x+x')dx e^{-j2\pi\omega x'} \, dx' \]

\[ = \int_{-\infty}^{\infty} g^*(x) \int_{-\infty}^{\infty} g(x+x')e^{-j2\pi\omega x'} \, dx \, dx' \]

\[ = G(\omega) \int_{-\infty}^{\infty} g(x)e^{-j2\pi\omega x'} \, dx' \]

\[ = G(\omega)G^*(\omega) = |G(\omega)|^2 \]
10 Discrete-time and periodic signals

The Fourier transform of a continuous-time aperiodic signal $g(x)$ is continuous and aperiodic.

What happens if the continuous signal is sampled at discrete times, or if the signal is periodic?

We need the notion of an impulse (Dirac distribution), defined by

$$g(x) = \int_{-\infty}^{\infty} g(x') \delta(x - x') dx'$$

and that of an impulse train ("shah"),

$$\mathbb{I}(x) := \sum_{k=-\infty}^{\infty} \delta(x - k\Delta x)$$

The impulse train is its own Fourier transform.

10.1 Fourier transform, Fourier series, DFT

Fourier transform

$$G(\omega) = \int_{-\infty}^{\infty} g(x) e^{-j2\pi \omega x} dx \quad \leftrightarrow \quad g(x) = \int_{-\infty}^{\infty} G(\omega) e^{j2\pi \omega x} d\omega$$

Fourier series (discrete-time Fourier transform, DTFT)

$$G(n) = \frac{1}{N} \int_{-N/2}^{N/2} g(x) e^{-j2\pi nx/N} dx \quad \leftrightarrow \quad g(x) = \sum_{n=-\infty}^{\infty} G(n) e^{j2\pi nx/N}$$

discrete Fourier transform (DFT)

$$G(n) = \frac{1}{N} \sum_{k=0}^{N-1} g(k) e^{-j2\pi nk/N} \quad \leftrightarrow \quad g(k) = \sum_{n=0}^{N-1} G(n) e^{j2\pi nk/N}$$
10.2 DFT

The discrete Fourier transform can be expressed as a matrix multiplication

\[ G = \frac{1}{N} W_N g \]

with

\[ [W_N]_{n,k} = e^{-j2\pi nk/N} = \left( e^{-j2\pi /N} \right)^{nk} \]

The inverse is given by

\[ g = W_N^* G \]

where \( W_N^* \) is the complex conjugate, not the adjoint:=transposed complex conjugate.