1 Hilbert transform – analytic wavelet – wavelet ridges

1.1 Overview

The Hilbert transforms make use of symmetries of the Fourier transform allowing for special spectral representations of signals.

- Analytic wavelets are some interesting representations obtainable using Hilbert transforms
- Wavelet ridges give estimates for spectral features

1.2 Common properties for Fourier transforms

1.2.1 Symmetry

Any signal $f \in \mathbb{C}$ can be decomposed into parts having
- conjugate symmetry ("even" symmetry for $f \in \mathbb{R}$)
- conjugate antisymmetry ("odd" symmetry for $f \in \mathbb{R}$)

$$f(x) = f_e(x) + f_o(x) \quad \text{with} \quad \begin{cases} f_e(x) = \frac{1}{2} (f(x) + f^*(-x)) = f^*_e(-x) \\ f_o(x) = \frac{1}{2} (f(x) - f^*(-x)) = -f^*_o(-x) \end{cases}$$

If $f \in \mathbb{R}$ we have for the Fourier transforms:

$$f(x) \leadsto F(e^{j\omega}) = F_{\mathbb{R}}(e^{j\omega}) + jF_{\mathbb{I}}(e^{j\omega})$$
$$f(x) = f^*(x) \leadsto F^*(e^{-j\omega}) = F_{\mathbb{I}}^*(e^{-j\omega}) - jF_{\mathbb{R}}^*(e^{-j\omega})$$

$$F_{\mathbb{R}}(e^{j\omega}) = F_{\mathbb{R}}(e^{-j\omega}) \implies F_{\mathbb{R}} \text{ is even},$$
$$F_{\mathbb{I}}(e^{j\omega}) = -F_{\mathbb{I}}(e^{-j\omega}) \implies F_{\mathbb{I}} \text{ is odd}.$$
Symmetry cont’d

Finally, also for $f \in \mathbb{R}$, $F_R$ and $F_I$ can be composed like:

$$F_R(e^{j\omega}) = \frac{1}{2}(F(e^{j\omega}) + F(e^{-j\omega}))$$

$$F_I(e^{j\omega}) = -\frac{j}{2}(F(e^{j\omega}) - F(e^{-j\omega}))$$

1.2.2 Causal signals

Def:

Causality:

$$f(x) = 0 \text{ for } x < 0$$

(alike for sequences)

Consequences:

A causal sequence can be recovered from only its even part (from the odd part only for $n > 0$).

Be $u[n]$ the unit step, $u[n \geq 0] = 1$ and $\delta[n]$ the Dirac pulse:

$$f[n] = 2f_o[n]u[n] - f_o[0]\delta[n]$$

$$f[n] = 2f_o[n]u[n] + f_o[0]\delta[n]$$

Note:

For discrete sequences of finite length $N$, the notion of causality is to be modified:

DFT requires periodic signals $\longrightarrow$ continuation:

$$\tilde{f}[n] = f[(n)_N]$$

Def:

Periodical causality for $f[n]$ periodic sequence:

$$\tilde{f}[n] = 0 \text{ for } \frac{N}{2} < n < N$$

1.3 Analytic vs. real wavelets

- analytic wavelets $\psi(t) \in \mathbb{C}$ (p. ex. Gabor-wavelet)
  separate amplitude and phase of a signal, measuring the instantaneous frequency
  ($\sim$ windowed Fourier transform)
- real wavelets $\psi(t) \in \mathbb{R}$ (p. ex. Mexican hat-wavelet)
  allow for sharp transient detection
1.4 Analytic functions

Def:

\[ G \subseteq \mathbb{C} : f \text{ analytic (regular, holomorph) in } G : f(z) \text{ differentiable } \forall z \in G \]

C-diff’able functions are smooth, analytic functions are therefore infinitely diff’able.

**Theorem:** Cauchy-Riemann:

\[ f(z = x + jy) = u(x,y) + jv(x,y) \text{ is diff’able if and only if for the smooth partials holds:} \]

\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

Then also holds (Laplace eqns):

\[ \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \]

The complex \( f(z) \) is (up to an additive \( j\phi(x) \)) determined by its real part \( u(x,y) \).

Integrating the CR-eqn for \( u(x,y) \) yields the imaginary counterpart \( v(x,y) \):

\[ v = -\int \frac{\partial u}{\partial x} dy + \phi(x) \quad \text{with} \quad \frac{d\phi}{dx} = -\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) \frac{\partial u}{\partial y} \]

The notion of analytic signals gives similar powerful possibilities . . .

1.5 Analytic signals

Def:

\( f_a \in L^2(\mathbb{C}) \) is called analytic, if for the Fourier transform holds:

\[ \hat{f}(\omega) = 0 \text{ for } \omega < 0 \]

To obtain the analytic part of a signal \( f(t) \), we therefore set in the Fourier domain:

\[ \hat{f}_a(\omega) = \begin{cases} 2\hat{f}(\omega) & \text{if } \omega \geq 0 \\ 0 & \text{if } \omega < 0 \end{cases} \]

and apply the inverse transform.

- for \( f \in L^2(\mathbb{R}) \), \( f_a \) is necessarily complex
- analytic signals fulfill the Cauchy-Riemann eqns
- the notion can straightforward be extended towards analytic sequences \( f_a[n] \)

1.6 Discrete Hilbert transform relations

Correlations between real and imaginary parts of Fourier transforms.

1.6.1 Real sequences

Let \( f[n] \) be a causal, stable and real sequence: \( f[n] \in \mathbb{R} \).

Apply convolution to the even part reconstruction \( f[n] - 2f_c[n]u[n] + f_c[0] \delta[n] \):

\[ f_c(\omega) = \frac{1}{\pi} \int_0^\infty f_c(\omega) d\omega = f(\omega - \theta) d\theta - \delta[0] \]

Fourier transform representation of unit step:

\[ \hat{u}(\omega) = \sum_{k=-\infty}^{\infty} \pi \delta(\omega - 2\pi k) + \frac{1}{1 - e^{-j\omega}} = \sum_{k=-\infty}^{\infty} \pi \delta(\omega - 2\pi k) + \frac{1}{2} - \frac{j}{2} \cot \left( \frac{\omega}{2} \right) \]
1.6.2 Complex sequences

Hilbert transformer relations can also be derived between real-parts and imaginary-parts of discrete complex sequences...

\[ f[n] = f_R[n] + j f_I[n] \rightarrow \hat{f}[n] = \hat{f}_R[n] + j \hat{f}_I[n] \]

The symmetry / anti-symmetry relations for \( \hat{f}_R, \hat{f}_I \) hold as before. We can reconstruct the full Fourier transform like

\[
\hat{f}[n] = \begin{cases} 
2 \hat{f}_R[n] & \text{for } 0 \leq \omega < \pi \\
0 & \text{for } -\pi \leq \omega < 0
\end{cases}
\]

Alternatively, we see

\[
\hat{f}_I[n] = \begin{cases} 
-j \hat{f}_R[n] & \text{for } 0 \leq \omega < \pi \\
\hat{f}_R[n] & \text{for } -\pi \leq \omega < 0
\end{cases}
\]

\[
H(\omega) = \begin{cases} 
-j & \text{for } 0 \leq \omega < \pi \\
j & \text{for } -\pi \leq \omega < 0
\end{cases} = H(\omega) f_R[n]
\]

\( H(\omega) \) is an ideal \( \frac{\pi}{2} \)-phase shifter, the Hilbert transformer:

\[
H(\omega) = \begin{cases} 
-j & \text{for } 0 \leq \omega < \pi \\
j & \text{for } -\pi \leq \omega < 0
\end{cases}
\]

Note: This ideal Hilbert transformer is non-causal, as

\[
H(\omega) = \sum_{n=-\infty}^{\infty} h[n] e^{-j \omega n} \quad \text{with impulse response } h[n] = \begin{cases} 
\frac{2 \sin^2(\frac{\omega n}{2})}{n} & n \neq 0 \\
0 & n = 0
\end{cases}
\]

\( \rightarrow \) approximate representations possible with discrete Kaiser-window FIR filters.

1.7 Instantaneous frequency

Def: \textit{Instantaneous frequency} \( \omega(t) \): derivative of generalized phase \( \phi(t) \)

\[
\omega(t) = \dot{\phi}(t) \text{ sign}(\dot{\phi}(t)) \geq 0
\]

Consider an amplitude-modulated signal:

\[
f(t) = a(t) \cos(\omega_d t + \phi_d)
\]

Having also time variations of the amplitude,

\[
f(t) = a(t) \cos \phi(t) \text{ with } a \geq 0
\]

\( \rightarrow a(t) \) and \( \phi(t) \) are no longer uniquely defined.

If we consider the analytic part of the signal, \( f_a \), written like

\[
f_a(t) = a(t) e^{i \phi(t)}
\]

we find a unique representation with
1.7.1 Application - frequency modulation

Frequency modulated signals are more robust against additive Gaussian white noise than amplitude modulations.

Encode message signal as \( m(t) \):

\[
f(t) = a \cos \phi(t) \quad \text{with} \quad \phi(t) = \omega_0 + km(t)
\]

Chosen bandwidth of \( f(t) \) is proportional to \( k \). Computation of instantaneous frequency \( \dot{\phi}(t) \) ("demodulation") directly restores \( m(t) \).

1.7.2 Example – representation of bandpass signals

Complex lowpasses come in handy as representations of narrow bandpass (communications) signals. Consider the analytic lowpass representation

\[
f[n] = f_{\Re}[n] + jf_{\Im}[n] \quad \text{Hilbert transform of} \quad f_{\Re}[n]
\]

and

\[
\hat{f}(\omega) = 0 \quad \text{for} \quad -\pi < \omega < 0
\]

We find a corresponding bandpass with the modulation

\[
s[n] := f[n]e^{j\omega_0 n}
\]

It can be shown that if \( \hat{f}(\omega) = 0 \text{ for } \Delta \omega < |\omega| \leq \pi \):

\[
\Rightarrow \quad s[n] \text{ is a single-sided bandpass: } s(\omega) \neq 0 \text{ only for } \omega_k < \omega \leq \omega_k + \Delta \omega.
\]

Alternative representation: analytic amplitude and instantaneous phase:

\[
f[n] = A[n]e^{j\phi[n]} \quad \rightarrow \quad s[n] = (f_{\Re}[n] + jf_{\Im}[n])e^{j\omega_0 n} = A[n]e^{j(\omega_0 n+\phi[n])}
\]

1.7.3 Sampling of bandpass signals

Consider a bandpass signal \( s[n] \) with Fourier transform

\[
S(\omega) \begin{cases} 
0 & \text{for } 0 \leq |\omega| \leq \omega_k \\
\neq 0 & \text{for } \omega_k < |\omega| < \omega_k + \Delta \omega \\
0 & \text{for } |\omega| \geq \omega_k + \Delta \omega
\end{cases}
\]

Straightforward, sampling at Nyquist rate \( T \) would require \( 2(\omega_k + \Delta \omega) = \frac{2\pi}{T} \) samples.

Use the Hilbert transformer to move the spectrum around zero:

1. \( s_{\Re}[n] = s[nT] \), and \( s_{\Im}[n] = h[n] \ast s[nT] \)

2. Compress signal by \( M = \lfloor \frac{\omega_k + \Delta \omega}{2\pi} \rfloor \) : \( s_c[n] = s[Mn] \)

Fourier transform is then:

\[
\hat{s}_c(\omega) = \frac{1}{M} \sum_{m=0}^{M-1} s(e^{j\omega 2\pi m}/M)
\]

3. Signal is now downsampled by \( M \), residing in the support interval: \(-\pi < \omega \leq \pi\)
1.7.4 Application and limits

Generic amplitude modulated signal $f(t) = a(t) \cos(\omega_0 t + \phi_0)$

Fourier transform:

$$\hat{f}(\omega) = \frac{1}{2} \left( \hat{a}(\omega - \omega_0) e^{i\phi_0} + \hat{a}(\omega + \omega_0) e^{-i\phi_0} \right)$$

If $a(t)$ is slow-varying compared to $a_0$, the second (high-frequency) term can be ignored. The analytic signal is then

$$\hat{f}_a(\omega) = \hat{a}(\omega - \omega_0) e^{i\phi_0} \Rightarrow f_a(t) = a(t) e^{i(\omega_0 t + \phi_0)}$$

Sum of two sines:

$$f(t) = a \cos(\omega_1 t) + a \cos(\omega_2 t)$$

$$\Rightarrow f_a(t) = a e^{i\omega_1 t} + a e^{i\omega_2 t}$$

$$= a \cos \left( \frac{1}{2} (\omega_1 - \omega_2) t \right) \exp \left( i \frac{1}{2} (\omega_1 + \omega_2) t \right)$$

We can see that

- instantaneous frequency $\dot{\phi}(t) = \frac{1}{2} (\omega_1 - \omega_2) \Rightarrow$ somehow an average
- analytic amplitude: $a(t) = R(\ldots) = a \left| \cos \left( \frac{1}{2} (\omega_1 - \omega_2) t \right) \right|$

Results:

- The instantaneous frequency can be used to analyze signals with time-varying amplitude,
- but: it cannot decompose multiple frequency components!

$\Rightarrow$ calculate windowed Fourier ridges, ridges of modulated wavelets

The short-time Fourier transform uses atoms of constant time and frequency resolution in the entire time-frequency plane.

The continuous wavelet transform uses at

- low frequencies: atoms with high frequency resolution and low spatial resolution
- high frequencies: atoms with low frequency resolution and high spatial resolution

Consequences for spectral analysis:

- windowed Fourier atoms can detect the frequency of slowly changing signals precisely, but cannot follow the instantaneous frequency of rapid variations due to their fixed scale
- wavelet atoms can follow rapid variations, but can cause interference of similar-frequency signals due to scale mismatch
1.8 Analytic wavelets

Analytic wavelets have the properties of analytic signals. Construction using frequency modulation of a real, symmetric window \( g(t) \):

\[
\psi(t) = g(t)e^{i\eta t} \quad \Rightarrow \quad \hat{\psi}(\omega) = \hat{g}(\omega - \eta)
\]

Analytic signal: \( \psi(\omega) = 0 \) for \( \omega < 0 \) \( \Rightarrow \) restriction \( \hat{g}(\omega) = 0 \) for \( |\omega| > \eta \)

\( g \) real, symmetric \( \Rightarrow \) \( \hat{g} \) real, symmetric (even)

Center frequency \( \tilde{\omega} \):

\[
\tilde{\omega}(0) = \sup_{\omega \in \mathbb{R}} |\hat{\phi}(\omega)| = |\hat{\phi}(\eta)| \quad \Rightarrow \quad \tilde{\omega} = \eta
\]

Example: Gabor wavelet ...

\[
\hat{g}(\omega) = \frac{1}{\sqrt{\pi \sigma^2}} e^{-\frac{\omega^2}{\sigma^2}}
\]

Fourier transform:

\[
\hat{g}(\omega) = \frac{1}{\pi \sigma^2} e^{-\frac{\omega^2}{\sigma^2}}
\]

Gabor wavelets can be approximately analytic:

Given an \( \eta \) with \( \sigma^2 \eta^2 \ll 1 \), then

\[
\hat{g}(\omega) \approx 0 \quad \text{for} \quad |\omega| > \eta
\]

Wavelet ridges – properties

Consider the normalized power spectrum ("scalogram"):

\[
\frac{1}{s} P_W^f(u, \eta_s) := \frac{|Wf(u, \eta_s)|^2}{s}
\]

with the wavelet transform result

\[
\frac{1}{s} P_W^f(u, \eta_s) = \frac{1}{4 \sigma^2} \left( \left| \hat{g} \left( \eta \left( 1 - s \phi'(u) \right) + \epsilon \left( u, \frac{\eta}{s} \right) \right) \right|^2ight)
\]

Around the maximum of \( \hat{g} \) at \( \omega = 0 \), neglect \( \epsilon \left( u, \frac{\eta}{s} \right) \).

Maximum of power spectrum is at:

\[
\phi'(u) = \frac{\eta}{s(u)} = : \xi(u)
\]

with the analytic amplitude

\[
a(u) = \frac{2 \sqrt{s} \frac{1}{s} P_W^f(u, \frac{\eta}{s})}{|\hat{g}(0)|}
\]

Def: Wavelet ridges: set of points \( (u, \xi(u)) \)
Error term $\varepsilon(u, \eta)$:

From the wavelet transform $Wf(u, s)$ we see that the complex phase is $\Phi_W(u, \eta) := \phi$.

At ridge points:

$$\frac{\partial \Phi_W(u, \eta)}{\partial u} = \phi'(u) = \xi(u)$$

The power spectrum is then

$$\frac{1}{s} P_W f(u, \eta) - \frac{1}{4} a^2(u) \left| \varepsilon(u, \eta) \right|^2$$

Exact calculations: $\varepsilon$ stays small if

$$\frac{\eta^2 \left| d'(u) \right|}{\phi'(u)^2 \left| a(u) \right|} \ll 1 \quad \eta^2 \left| \phi''(u) \right| \phi'(u)^2 \ll 1$$

Therefore: If the instantaneous frequency $\phi'(u)$ is small, the analytic amplitude $a'(u)$ and $\phi'(u)$ may vary only slowly.

1.8.3 Comparison: Windowed Fourier vs. Wavelet ridges

Consider the sum of two parallel linear chirps:

$$f(t) = a_1 \cos(b_1 t^2 + c_1) + a_2 \cos(b_2 t^2).$$

Example – hyperbolic chirp

This signal is emitted by radars, sonar devices and bats...
Windowed-Fourier spectrogram, wavelet scalogram and detected ridges

Windowed-Fourier transform

Gabor-wavelet transform

Thanks for the patience!