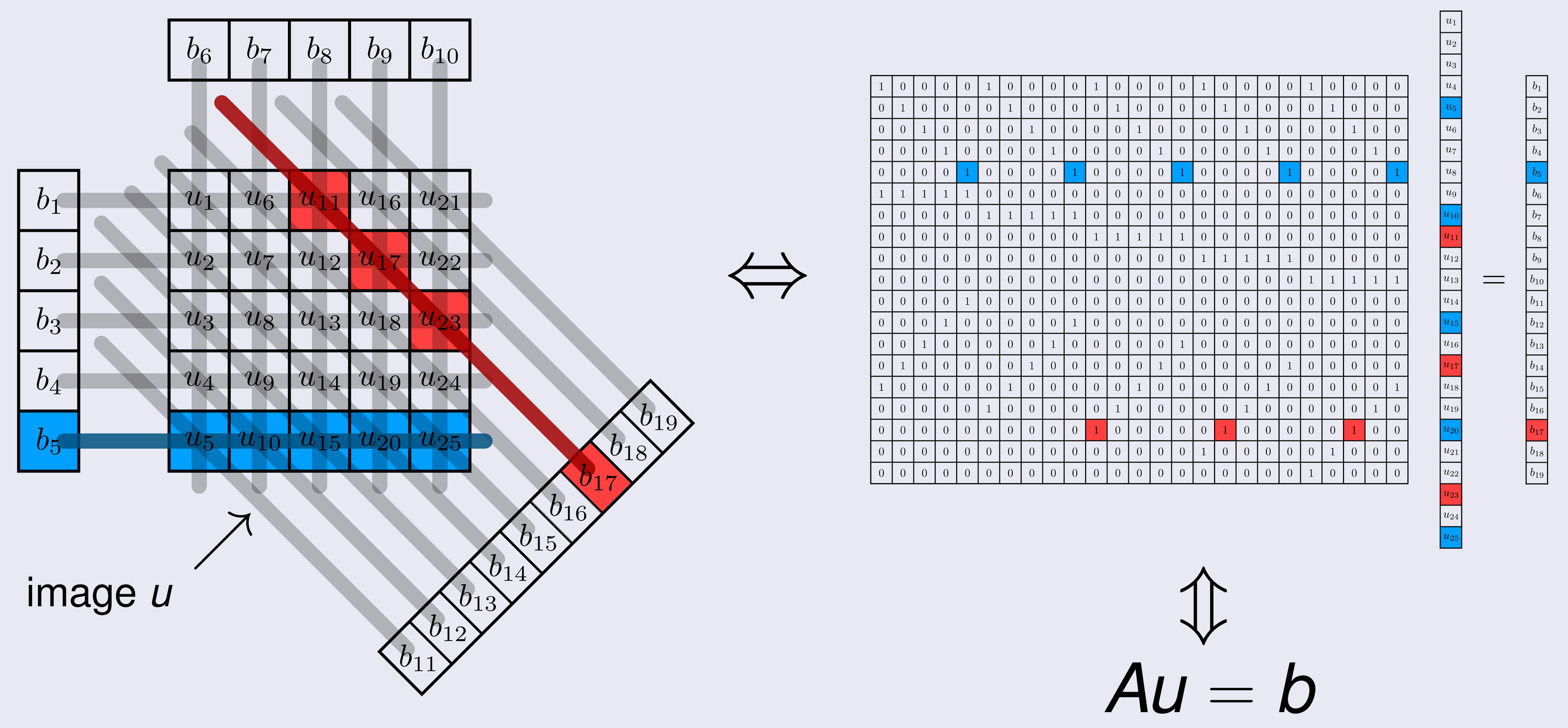
On Compressed Motion Sensing for Tomographical Particle Image Velocimetry

Robert Dalitz, Ecaterina Bodnariuc, Stefania Petra, Christoph Schnörr
Mathematical Imaging Group, Heidelberg University, Germany
e-mail: {robert.dalitz,ecaterina.bodnariuc}@iwr.uni-heidelberg.de, {petra,schnoerr}@math.uni-heidelberg.de

Abstract

Aim: Recovery of images and estimation of the flow
Starting point: Particle Image Velocimetry ("*PIV*"), i. e. standard tomographic recovery followed by cross correlation
Approach: Extend standard tomographic sensor by information of transformed image
Result: Extended sensors show significantly improved recovery performance

Standard Tomographic Recovery (existing)



Given: Sensor $A \in \mathbb{R}^{m\times n}$, observations $b \in \mathbb{R}^m$ with

$$b_i = \int_{\mathcal{R}_i} u(x)dx \approx \sum_{j=1}^{n} u_j \underbrace{\int_{\mathcal{R}_i} \mathcal{P}_j(x)dx}_{=:a_{ij}} \qquad (1)$$

where entry a_{ij} in A is the length of the intersection of ray $\mathcal{R}_i$ with pixel $\mathcal{P}_j$.
Wanted: Original particle image $u \in \mathbb{R}^n$
Task: Problem highly underdetermined
But accurate recovery possible, if u sufficiently sparse [5]
Relaxed Solution: Solve $Au \approx b$ via

$$\min_{u\geq 0} \|Au - b\|^2$$

Classic Particle Image Velocimetry (PIV) (existing)

1. Recover two images u, u_t (sparsity desired) for two points in time via
$$\min_{u\geq 0} \|Au - b\|^2 \quad \text{and} \quad \min_{u_t\geq 0} \|Au_t - b_t\|^2$$
2. Use *cross-correlation* [4] to estimate transport (high particle density desired $\Leftrightarrow$ u less sparse)
$$T_t(u) = u_t$$

Compressed Motion Sensing (our approach)

Recover u and use available information at time step t. Consider projections b_t as additional measurement together with b:

$$\min_{u\geq 0} \|Au - b\|^2 + \|\underbrace{AT_t(u)}_{\text{additional sensor}} - b_t\|^2 \quad \Leftrightarrow \quad \min_{u\geq 0} \left\| \underbrace{\begin{pmatrix} A \\ AT_t(\cdot) \end{pmatrix}}_{\text{complemented sensor}} u - \begin{pmatrix} b \\ b_t \end{pmatrix} \right\|^2$$

Key question: Assuming T_t is known, how much improves the recovery performance of the complemented sensor

$$A_T := \begin{pmatrix} A \\ AT_t(\cdot) \end{pmatrix} \ ? \qquad (2)$$

Remarks:

- We call compressed sensing in connection with the correspondence information $u_t = T_t(u)$ *compressed motion sensing*.
- The problem of jointly estimating the images and the transformation parameters from the available multi-view measurements is not addressed by this work, but is subject of our current research and is a well studied topic outside the compressed sensing literature [7].

Theoretical Case Study: Poiseuille Flow in a Pipe

- Compressed sensing scenario relevant to blood flow estimation [3]
- At time-step $\tau \in [0, t]$ we consider the particle function $u_\tau : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$
$$u_\tau(x) = \sum_{k=1}^{s} \exp\left(-\frac{\|x - p_\tau^k\|^2}{2\sigma^2}\right),$$
where particles are
  - centered at
  $$p_\tau^k = (p_{\tau,1}^k, p_{\tau,2}^k) \in \mathbb{R} \times [-r, r]$$
  - transported according to the Poiseuille law
  $$p_{\tau,1}^k = p_{0,1}^k + \tau v(p_{0,2}^k), \qquad p_{\tau,2}^k = p_{0,2}^k,$$
  with velocity profile
  $$v(z) = v_m\left(1 - z^2/r^2\right), \ z \in [-r, r],$$
  parametrised by the maximal velocity v_m and the tube's radius r.
- With $u := u_0$ this defines a corresponding transform $T_t(u) = u_t$ and the compressed motion sensor A_T by (2).
- The compressed motion sensor A_T is known, if the parametrized motion field can be estimated.

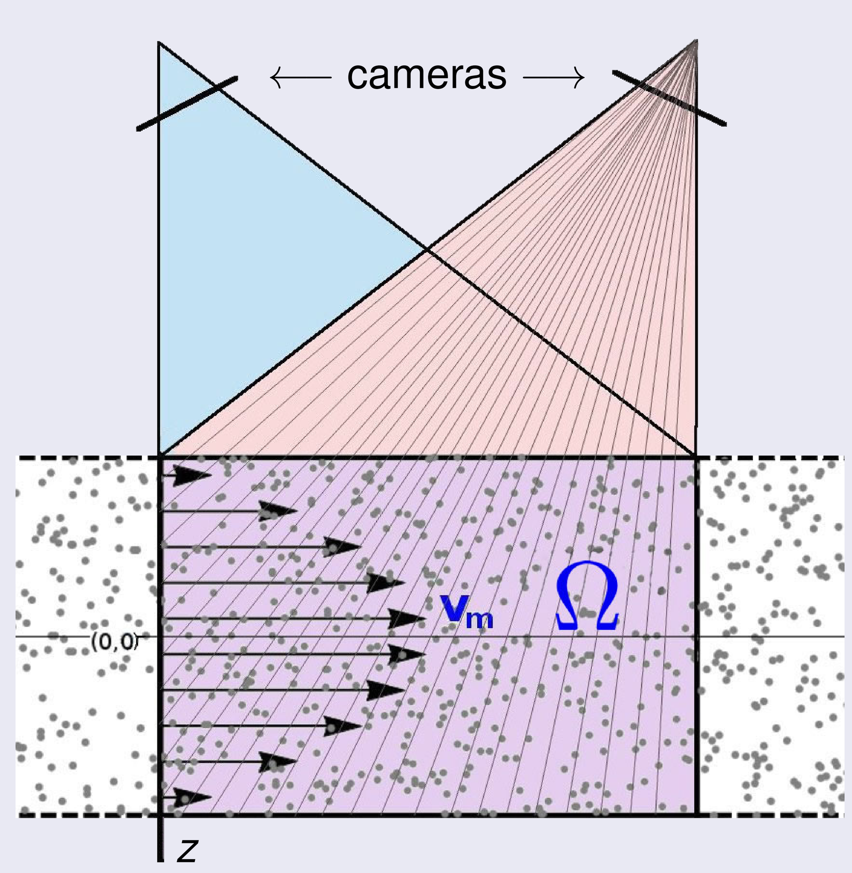


Figure: Imaging setup. For projection geometry used, see [6].

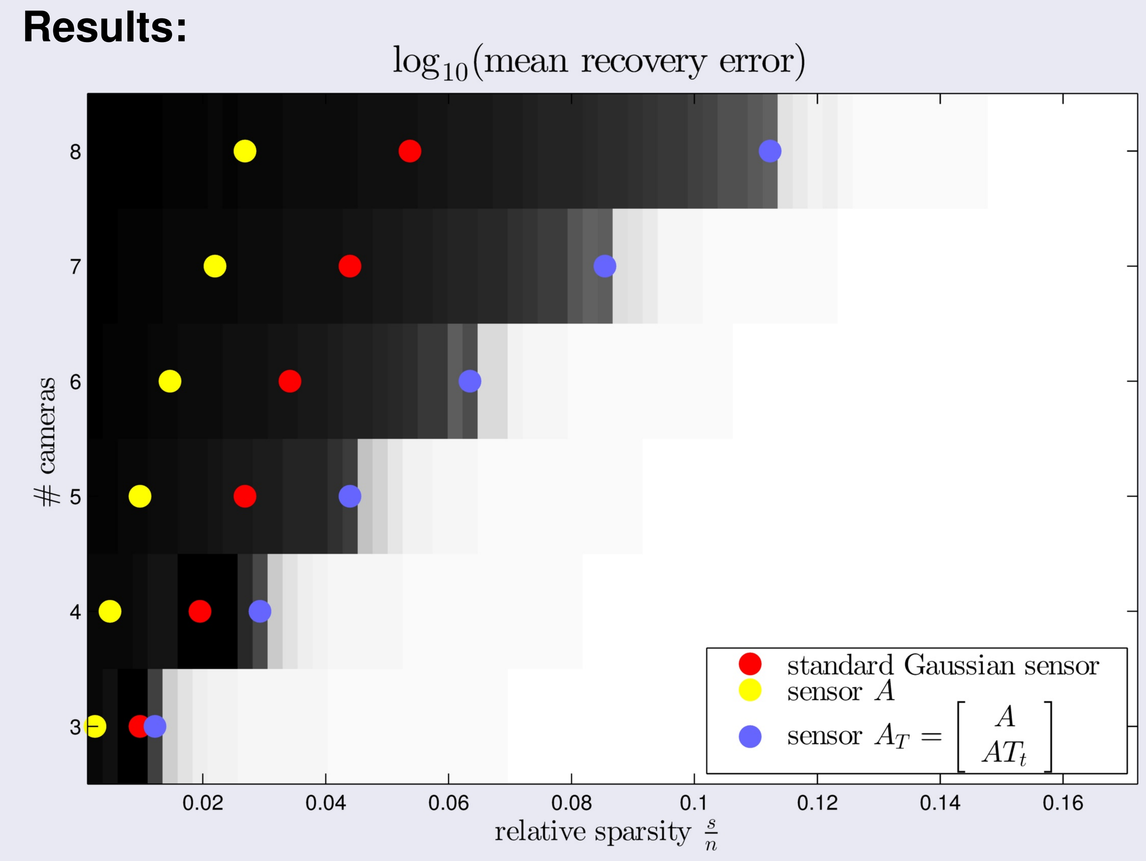
- The above definition of T_t allows us to investigate whether its inverse exists for the considered image class, and whether it is bi-Lipschitz [1].

Experiment: Discrete Poiseuille Flow

Numerical evaluation of sensor A vs. A_T for the Poiseuille flow scenario (see above) in a discrete setting, i. e.:

- Rectangular Ω covered by regular pixel grid.
- Components of $u, u_t \in \mathbb{R}^n$ are indexed on the pixel grid.
- Sensor $A \in \mathbb{R}^{m\times n}$ measures line integrals of both u and u_t as in (1) and conforms to optical tomography as used in optical PIV [8].
- Discrete version of $T_t(\cdot)$ $\Rightarrow$ can be expressed with the help of a matrix T_t so that
$$T_t(u) = T_t u = u_t \Rightarrow \text{Sensor } A_T = \begin{bmatrix} A \\ AT_t \end{bmatrix} \in \mathbb{R}^{2m\times n}.$$
- Recovery by standard non-negative linear least squares algorithm
- Repeated 25 times for each s and each number of cameras
- Standard Gaussian sensor $A \in \mathbb{R}^{m\times n}$, with $a_{ij} \sim \mathcal{N}(0, 1)$ for comparison. Tabulated values provided by Jared Tanner [9], see also [2].

Results:



- The compressed motion sensor A_T significantly outperforms the performance of the poor TomoPIV sensing matrix A.
- A_T successfully recovers more particles for 4 or more cameras, compared to doubling the number of line integrals, or simply doubling the number of cameras.

- When using $2m$ Gaussian measurements higher particle densities can be correctly recovered as compared to A_T. However, A_T has the advantage of being sparse, deterministic and based on a real sensor.

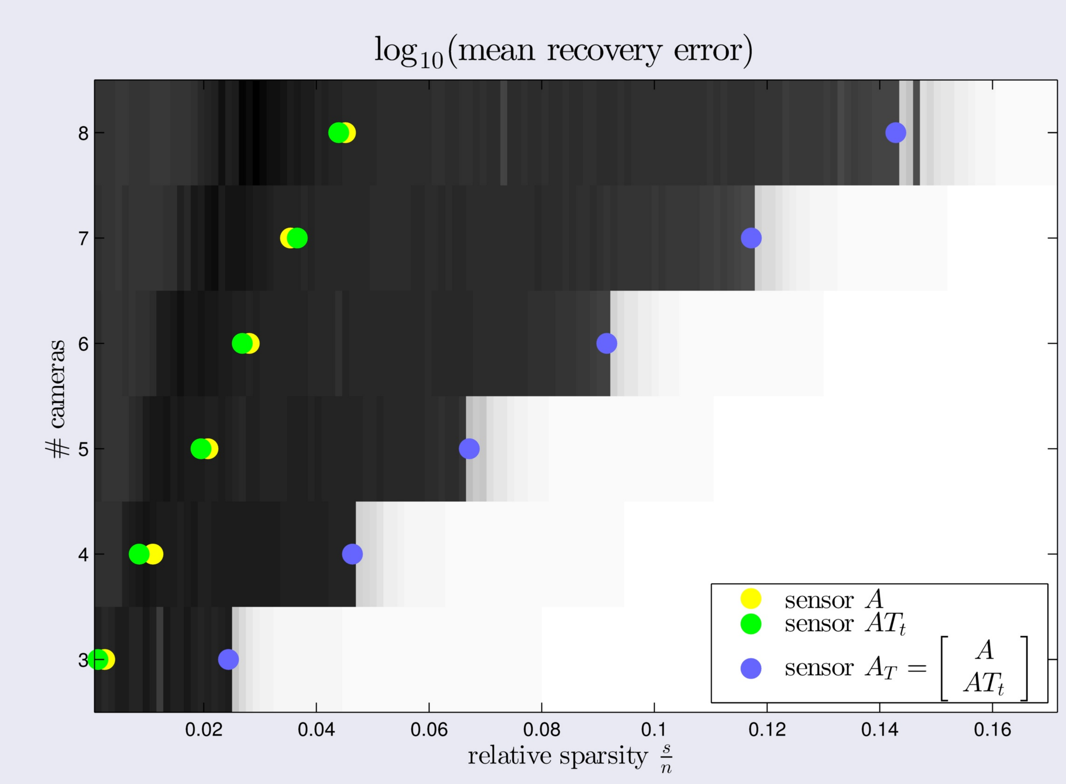
Experiment: Permutation as Transformation

Numerical evaluation of sensor A vs. A_T for T_t chosen as a random permutation matrix:

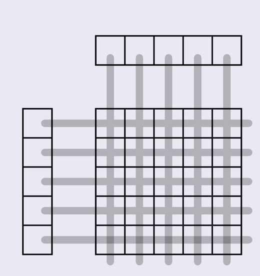
- Square grid and pixels.
- Components of $u, u_t \in \mathbb{R}^n$ are indexed on the pixel grid.
- Sensor $A \in \mathbb{R}^{m\times n}$ measures line integrals of both u and u_t as in (1).
- Sensor $T_t(\cdot) = T_t$ is a random permutation matrix.
- Recovery by standard non-negative linear least squares algorithm
- Repeated 100 times for each s and each number of cameras with a new random permutation sensor each time

Results:

- Again, the compressed motion sensor A_T outperforms the TomoPIV sensing matrix A.
- For 4 or more cameras: Again, A_T recovers more particles, compared to doubling the information available.
- A_T performes better than the Poiseuille flow sensor.



Example: 2 Cameras and Permutation as Transformation



- 2 orthogonal projections and $T_t(\cdot) = T_t$ is a random permutation matrix
- With just sensor $A \in \mathbb{R}^{m\times n}$, only one particle is guaranteed to be recovered.
- For sensor $A_T \in \mathbb{R}^{2m\times n}$ and a 10×10 image,
$$1 \leq \frac{(2m)_{\text{red}}(s)}{n_{\text{red}}(s)}$$
for $s \leq 11$ $\Rightarrow$ 11 particles guaranteed to be reconstructed [5].

Conclusion

- We empirically showed for a specific, but practically relevant, scenario that complementing the standard tomographic PIV sensor with a motion sensor, based on projections taken at a subsequent point of time, significantly improves recovery performance, provided motion can be estimated.
- Our current work on *compressed motion sensing* concerns generalizations of the Restricted Isometry Property for nonlinear sensors [1] as well as the joint problem of tomographic image reconstruction and motion estimation.

References

[1] Thomas Blumensath.
Compressed sensing with nonlinear observations and related nonlinear optimization problems.
Information Theory, IEEE Transactions on, 59(6):3466–3474, 2013.

[2] J. Tanner D.L. Donoho.
Precise undersampling theorems.
Proc. IEEE, 98:913–924, 2010.

[3] S. Petra E. Bodnariuc, A. Gurung and C. Schnörr.
Adaptive dictionary-based spatio-temporal flow estimation for echo piv.
Proc. EMMCVPR, LNCS Vol. 8932, 2015.

[4] S. Wereley J. Kompenhans M. Raffel, C. Willert.
Particle image velocimetry: A practical guide.
Springer, 2007.

[5] S. Petra and C. Schnörr.
Average case recovery analysis of tomographic compressive sensing.
Linear Algebra and its Applications, 441:168–198, 2014.
Special issue on Sparse Approximate Solution of Linear Systems.

[6] S. Petra, C. Schnörr, and A. Schröder.
Critical parameter values and reconstruction propertiesof discrete tomography: Application to experimental fluid dynamics.
Fundamenta Informaticae, 125:285–312, 2013.

[7] G. Puy and P. Vandergheynst.
Robust image reconstruction from multi-view measurements.
SIAM Journal on Imaging Sciences, 7(1):128–156, 2007.

[8] J. Westerweel C. Poelma R. Trip, D.J. Kuik.
An experimental study of transitional pulsatile pipe flow.
Physics of Fluids, 24, 2012.

[9] J. Tanner.
Restricted isometry constants.
http://people.maths.ox.ac.uk/tanner/ric_bounds.shtml.