Game Theory
Non-cooperative games

Speaker:
Philipp de Sombre
Outline

• Why should I care about Game Theory?
• How can I use Game Theory?
• What did I learn?
Why should I care about Game Theory?
Motivation

Mathematically appealing:
• Exciting thought processes
• Interesting solution concepts
• Different perspective on problems

Real-world implications:
• Problems can be modeled as games
• Used in a lot of fields
Real-world examples

Buying a car [3]  
-> Personal application

Environment [4]  
-> Public goods problem

Arms race [5]  
-> Dilemma situation

Understanding is the first step of solving.
How can I use Game Theory?
Fundamentals – Research fields

**Normative**

*What would be the best thing to do?*
- Rational Agent
- Mathematics, Economy

**Descriptive**

*What do people actually do?*
- Real People
- Psychology, Social sciences
Fundamentals – A Game

• A strategic game is a model of interacting decision-makers. [...] The model captures interaction between the players by allowing each player to be affected by the actions of all players, not only her own action. Each player has preferences about the action profile [...]. ([1] p. 11)

\[
\text{Game} = \{\text{Players}, \text{Strategies}, \text{Utilities}\}
\]

• What examples can you think of?
Normal form games - Assumptions

Static:
- Actions are “simultaneous”

Complete Information:
- All players know “everything”
Normal form games - elements

\[ G = \{ P, S, U \} \]

Players:
\[ P = \{1, 2, ..., n\} \]

Pure Strategies:
\[ S = \{S_1, S_2, ..., S_n\} \]

Payoff functions:
\[ U = \{u_1, u_2, ..., u_n\} \]
Normal form games - elements

Pure Strategies:

\[ S = \{S_1, S_2, ..., S_n\} \]
\[ S_i = \{1, 2, ..., k\} \]
\[ \hat{s} = (s_1, s_2, ..., s_n) \]
\[ s_i \in S_i \]

Payoff functions:

\[ U = \{u_1, u_2, ..., u_n\} \]
\[ u_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R} \]
## Normal form games - payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>Player B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Even</td>
<td>Odd</td>
</tr>
<tr>
<td>Player A</td>
<td>Even</td>
<td>1,0</td>
</tr>
<tr>
<td></td>
<td>Odd</td>
<td>0,1</td>
</tr>
</tbody>
</table>

### Notes
- **A** wins when sum is even
Games

• https://aifg.desomb.re
## Game 0: Even-Odd

<table>
<thead>
<tr>
<th></th>
<th>Even</th>
<th>Odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>1,0</td>
<td>0,1</td>
</tr>
<tr>
<td>Odd</td>
<td>0,1</td>
<td>1,0</td>
</tr>
</tbody>
</table>

What did you choose and why?
**Game 1: Rock, Paper, Scissor**

<table>
<thead>
<tr>
<th>Player A</th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rock</td>
</tr>
<tr>
<td>Rock</td>
<td>1,1</td>
</tr>
<tr>
<td>Paper</td>
<td>3,0</td>
</tr>
<tr>
<td>Scissor</td>
<td>0,3</td>
</tr>
</tbody>
</table>

What did you choose and why?
# Game 2: Shared fridge

<table>
<thead>
<tr>
<th>Player A</th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low Quality</td>
</tr>
<tr>
<td>Low Quality</td>
<td>1,1</td>
</tr>
<tr>
<td>High Quality</td>
<td>0,2</td>
</tr>
</tbody>
</table>

What did you choose and why?
Best response

Idea:
What is the best thing to do for a given strategy of the opponent?

Definition:
The strategy $s_i \in S_i$ is player $i$’s best response to his opponents’ strategies $s_{-i} \in S_{-i}$ if:

$$u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i}) \forall s_i' \in S_i$$

([2], p. 70)
Best response

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \forall s'_i \in S_i$$
Game 3: Prisoner's dilemma

<table>
<thead>
<tr>
<th>Player A</th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Say nothing</td>
</tr>
<tr>
<td>Say nothing</td>
<td>-1,-1</td>
</tr>
<tr>
<td>talk</td>
<td>0,-3</td>
</tr>
</tbody>
</table>

What did you choose and why?
Dominated strategies

Idea:

Is there a strategy that's always better than the others?

Definition:

Let $s_i \in S_i$ and $s'_i \in S_i$ be possible strategies for player $i$. We say that $s'_i$ is dominated by $s_i$ if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, player $i$'s payoff from $s'_i$ is less than that from $s_i$.

$$u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \forall s_{-i} \in S_{-i}$$

$\geq$: Weakly dominated  $>$: Strictly dominated  ([2], p. 60 ff)
Dominated strategies

\[ u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i}) \quad \forall s_{-i} \in S_{-i} \]

<table>
<thead>
<tr>
<th></th>
<th>Say nothing</th>
<th>talk</th>
</tr>
</thead>
<tbody>
<tr>
<td>Say nothing</td>
<td>-1, -1</td>
<td></td>
</tr>
<tr>
<td>talk</td>
<td>0, -3</td>
<td>-2, -2</td>
</tr>
</tbody>
</table>

'is dominated' \(\Rightarrow\) irrational player will never play it
# Game 4: Diner’s dilemma

<table>
<thead>
<tr>
<th>Player A</th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expensive Meal</td>
<td>Expensive Meal: 2,2</td>
</tr>
<tr>
<td></td>
<td>Cheap Meal: 3,0</td>
</tr>
<tr>
<td>Cheap Meal</td>
<td>Expensive Meal: 0,3</td>
</tr>
<tr>
<td></td>
<td>Cheap Meal: 2,2</td>
</tr>
</tbody>
</table>

What did you choose and why?
The Nash equilibrium

• Solution of a non-cooperative game
• Stable strategy profile
  -> No motivation for deviation for players
• may appear non-rational
  e.g.: Prisoner's dilemma
• Proof of existence for finite games by John Nash (1950)
The Nash equilibrium

Definition:
A Nash equilibrium is a strategy profile $\bar{s}^*$ with the property that no player $i$ can do better by choosing an action different from $s_i^*$, given that every other player $-i$ adheres to $s_{-i}^*$.

$$u_i(\bar{s}^*) \geq u_i(s_i, s_{-i}^*) \forall s_i \in S_i, \forall i \in P$$

([1], p. 20 ff)
Finding the Nash equilibrium

\[ u_i(s^*) \geq u_i(s_i, s^*_{-i}) \forall s_i \in S_i, \forall i \in P \]

<table>
<thead>
<tr>
<th>Player A</th>
<th>Expensive Meal</th>
<th>Cheap Meal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expensive Meal</td>
<td>2,2</td>
<td></td>
</tr>
<tr>
<td>Cheap Meal</td>
<td>0,3</td>
<td>2,2</td>
</tr>
</tbody>
</table>

Player B

Lacks a Nash equilibrium - Def. does not hold true.
### Game 5: Stag hunt

<table>
<thead>
<tr>
<th></th>
<th>Player A</th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Stag</td>
<td>Hare</td>
</tr>
<tr>
<td>Stag</td>
<td>3,3</td>
<td>0,2</td>
</tr>
<tr>
<td>Hare</td>
<td>2,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

What did you choose and why?

$\Rightarrow 2 \text{ Nash eq.}$
# Game 6: Partner work

<table>
<thead>
<tr>
<th>Player A</th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Word hard 2,2</td>
<td>Work hard 0,3</td>
</tr>
<tr>
<td>Goof off 3,0</td>
<td>Goof off 1,1</td>
</tr>
</tbody>
</table>

What did you choose and why?

Nash eq.: *but not the “best” solution for all players*
# Game 7: Marble game

<table>
<thead>
<tr>
<th></th>
<th>Player A</th>
<th></th>
<th>Player B</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Red</td>
<td>Blue</td>
<td>Yellow</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Red</td>
<td>2,2</td>
<td></td>
<td></td>
<td>0,3</td>
<td>0,1</td>
</tr>
<tr>
<td>Blue</td>
<td>3,0</td>
<td></td>
<td></td>
<td>2,2</td>
<td>2,0</td>
</tr>
<tr>
<td>Yellow</td>
<td>1,0</td>
<td></td>
<td></td>
<td>0,2</td>
<td>1,1</td>
</tr>
</tbody>
</table>

What did you choose and why?

Nash eq. (1) dominated by Blue
## Game 8: Battle of sexes

<table>
<thead>
<tr>
<th></th>
<th>Player B</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Football</td>
<td>Rom-com</td>
</tr>
<tr>
<td>Player A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Football</td>
<td>1,2</td>
<td>0,0</td>
</tr>
<tr>
<td>Rom-com</td>
<td>0,0</td>
<td>2,1</td>
</tr>
</tbody>
</table>

What did you choose and why?

\[\text{\textit{\& Nashq. in pure stats \& mixed stats.}}\]
What did I learn?
Further topics

2
3

Mixed strategies
-\(\Rightarrow\) Probability

Sequential games
-\(\Rightarrow\) Not simultaneous

Incomplete/Imperfect information
-\(\Rightarrow\) Uncertainty

Cooperative games
-\(\Rightarrow\) Contracts
-\(\Rightarrow\) Solution for dilemmas
Summary

Overview:
• Research fields
• Definition of a Game

Elements:
• Players
• Pure strategies
• Payoffs

Solution concepts:
• Best response
• Dominated Strategies
• Nash equilibrium
Questions
Results of the Games

https://aifg.desomb.re/#/scoreboard
Sources - Literature


Sources - Images

• John Nash: https://timedotcom.files.wordpress.com/2015/05/john-nash-2.jpg?quality=85
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