A Study of Lagrangean Decompositions and Dual Ascent Solvers for Graph Matching

Paul Swoboda[1] Carsten Rother[2] Hassan Abu Alhaija[2] Dagmar Kainmueller[3] Bogdan Savchynskyy[2]
[1] IST Austria [2] TU Dresden, Germany [3] MPI CBG, Germany
e-mail: pswoboda@ist.ac.at

Goal and contribution

Goal: We study efficient Lagrangean decomposition based algorithms for graph matching and propose new state-of-the-art ones.

Contribution:

- Several Lagrangean decompositions of the graph matching problems:
  1. **as a CRF**
  2. **with a network flow subproblem**
  3. **with additional label factors**
- Associated dual block coordinate ascent algorithms.
- Extensive numerical comparison against leading Lagrangean decomposition based solvers [4, 5].

Practical impact:

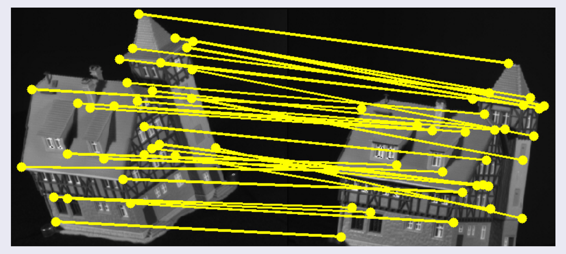
- Test data and C++ implementation available at `https://github.com/pawelswoboda/LP_MP`.

Theoretical foundations of used algorithms [3].

Graph matching problem

- Graph $G = (V, E)$.
- Discrete universe of labels $\mathcal{L}$.
- Discrete label space $X = \prod_{u \in V} X_u$ with $X_u \subseteq \mathcal{L}$.
- Unary costs $\theta_u : X_u \to \mathbb{R}$ for $u \in V$.
- Pairwise costs $\theta_{uv} : X_u \times X_v \to \mathbb{R}$ for $uv \in E$.

$$\underbrace{\min_{x \in X} \sum_{u \in V} \theta_u(x_u) + \sum_{uv \in E} \theta_{uv}(x_{uv})}_{\text{CRF}} \quad \text{s.t.} \quad \underbrace{x_u \neq x_v \quad \forall u \neq v}_{\text{uniqueness constraints}}$$



1) Transformation to CRF

Modify pairwise potentials:

- Penalize same label for different nodes.

For each $u, v \in V$ with $X_u \cap X_v \neq \emptyset$:

- Set $\theta_{uv}(x, x) = \infty \; \forall x \in X_u \cap X_v$.

Example: $uv \in E$, $X_u = \{1, 2, 3, 4\} = X_v$. $\theta_{uv} =$

∞	$\theta_{uv}(1,2)$	$\theta_{uv}(1,3)$	$\theta_{uv}(1,4)$
$\theta_{uv}(2,1)$	∞	$\theta_{uv}(2,3)$	$\theta_{uv}(2,4)$
$\theta_{uv}(3,1)$	$\theta_{uv}(3,2)$	∞	$\theta_{uv}(3,4)$
$\theta_{uv}(4,1)$	$\theta_{uv}(4,2)$	$\theta_{uv}(4,3)$	∞

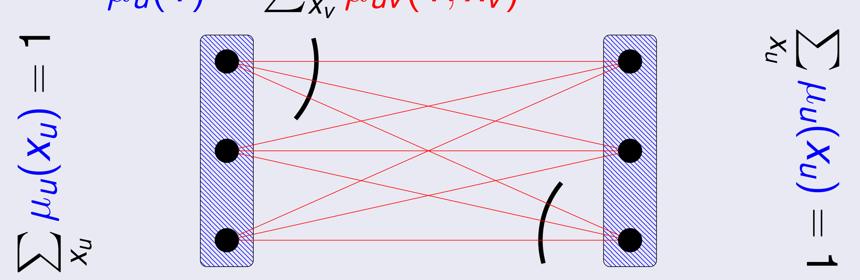
LP-relaxation for CRF

Optimization over **local polytope**: Simplex $\Delta_n = \{x \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$.

$$\min_{\mu \in L_G} \langle \theta, \mu \rangle := \sum_{u \in V} \langle \theta_u, \mu_u \rangle + \sum_{uv \in E} \langle \theta_{uv}, \mu_{uv} \rangle$$

$$L_G = \begin{cases} \mu_u \in \Delta_{|X_u|}, & u \in V, \\ \mu_{uv} \in \Delta_{|X_{uv}|}, & uv \in E, \\ \sum_{x_v \in X_v} \mu_{uv}(x_u, x_v) = \mu_u(x_u), & uv \in E, x_u \in X_u. \end{cases}$$

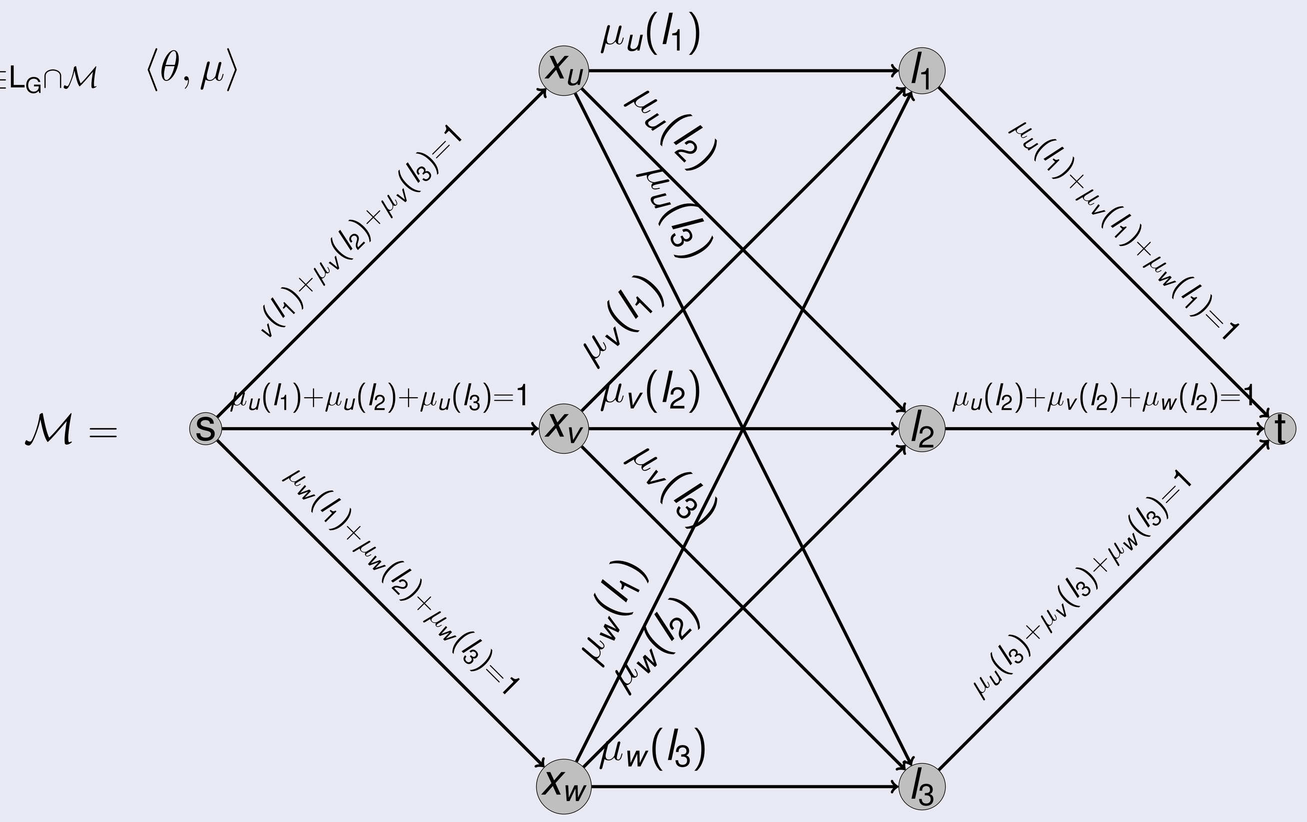
Example: constraints between unary and pairwise variables.



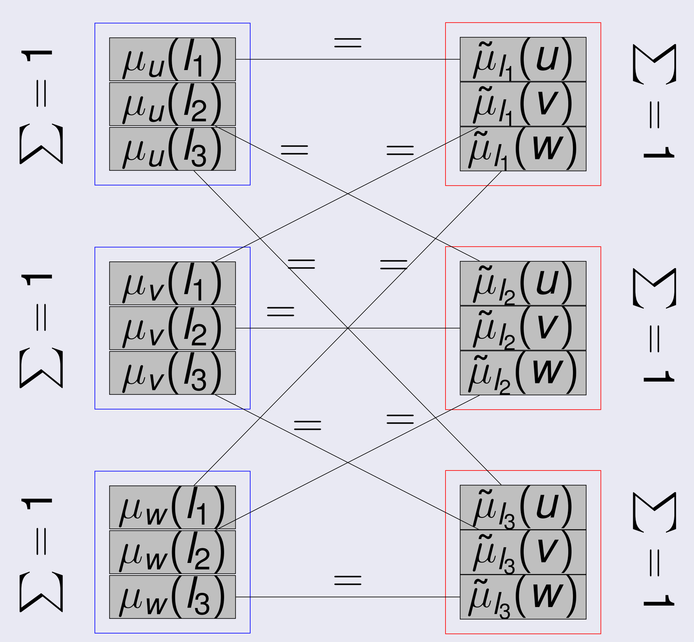
2) With underlying network flow subproblem

Model uniqueness constraint $x_u \neq x_v \; \forall u \neq V$ as network flow problem:

$$\min_{\mu \in L_G \cap \mathcal{M}} \langle \theta, \mu \rangle$$



3) With label factors



- Label factor for each label $l \in \mathcal{L}$.
- Each label factor forces label to be taken at most once.

Optimization problem:

$$\min_{\mu \in L_G, \tilde{\mu}} \langle \theta, \mu \rangle$$
$$\mu \in L_G$$
$$\text{s.t.} \quad \tilde{\mu}_s \in \text{conv}(X_s), \; s \in \mathcal{L}$$
$$\mu_u(s) = \tilde{\mu}_s(u), \; s \in X_u.$$

Optimization

- **Lagrangean decomposition, constraint dualization**:
  1. **CRF**: dualize constraints between unary/pairwise variables.
  2. **Network flow**: additionally dualize constraints between unary/network flow variables.
  3. **Label factor**: additionally dualize constraints between unary/label factors.
- **Optimize** with dual block coordinate ascent [3].
  1. **CRF**: dual block coordinate ascent TRWS [1].
  2. **Network flow**: augment TRWS [1] by passing messages to network flow factor $\mathcal{M}_G$.
  3. **Label factor**: augment TRWS [1] by passing messages to and from label factors.
- **Round primal solution**
  1. **CRF**: via dynamic programming.
  2. **Network flow**: use matching from network flow subproblem.
  3. **Label factor**: via dynamic programming.

Higher order extension

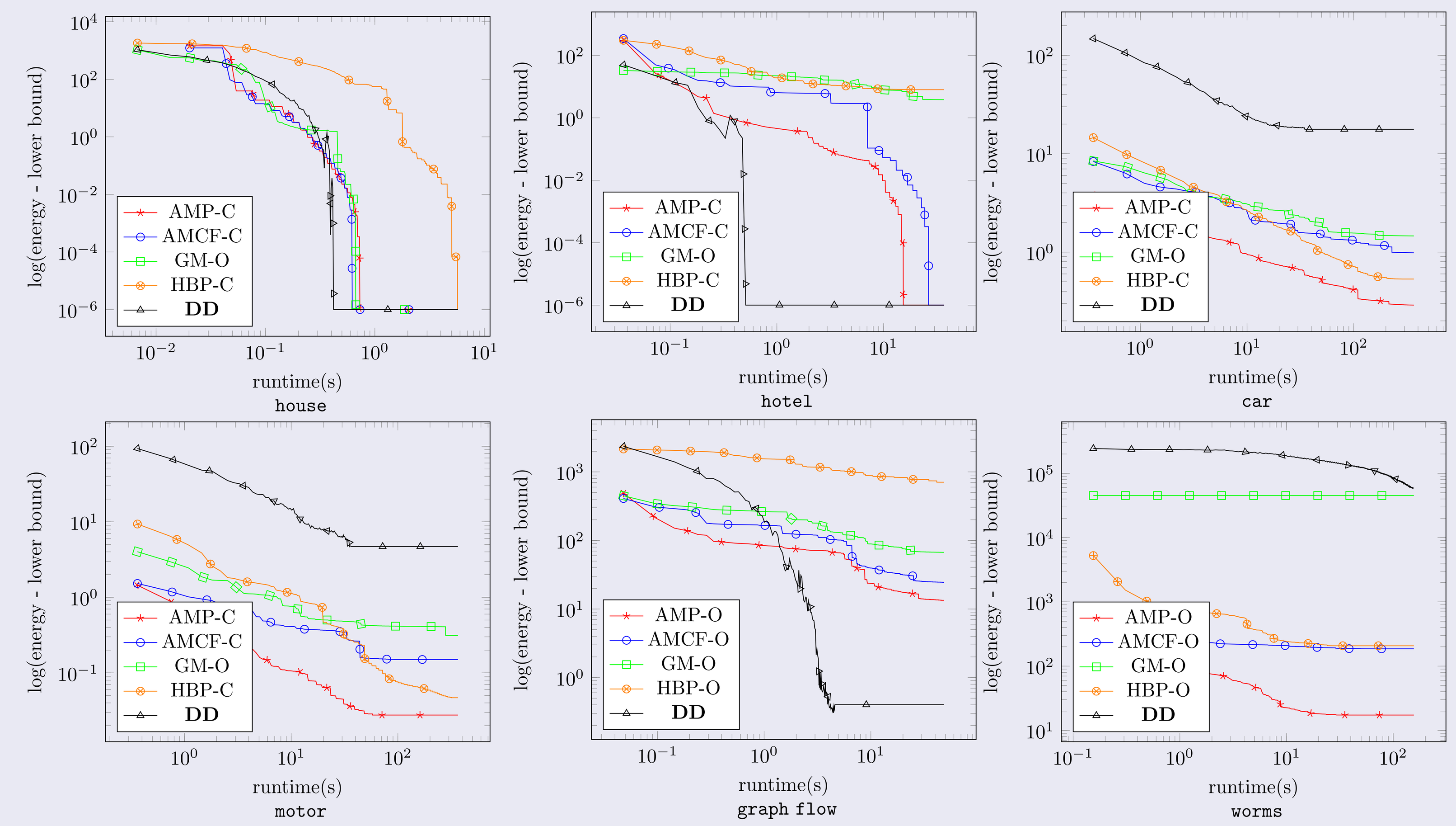
- Triplets $T \subset \binom{V}{3}$.
- Ternary costs $\theta_{uvw} : X_u \times X_v \times X_w \to \mathbb{R}$ for $uvw \in T$.

Higher order graph matching problem:

$$\min_{x \in X} \sum_{u \in V} \theta_u(x_u) + \sum_{uv \in E} \theta_{uv}(x_{uv}) + \sum_{uvw \in T} \theta_{uvw}(x_{uvw}) \quad \text{s.t.} \quad x_u \neq x_v \quad \forall u \neq v$$

- Relaxations and optimzers can be easily adapted to higher order.
- We use triplets for tightening relaxation [2].

Experimental results: averaged log(gap) vs runtime plots



- Our solvers:
  1. **CRF**: GM [1].
  2. **Network flow**: AMCF.
  3. **Label factors**: AMP.
- Competing solvers: DD [4], HBP [5]

Experimental results: averaged dual lower bound, primal energy, runtime

Dataset			/Algorithm	**AMP**	**AMCF**	**GM** [1]	**HBP** [5]	**DD** [4]
hotel	#I	105	LB	**-4293.00**	**-4293.00**	-4293.94	-4294.98	**-4293.00**
	#V	30	UB	**-4293.00**	**-4293.00**	-4290.12	-4287.00	**-4293.00**
	#L	30	time(s)	3.07	4.31	14.91	16.76	**0.42**
house	#I	105	LB	**-3778.13**	**-3778.13**	**-3778.13**	**-3778.13**	**-3778.13**
	#V	30	UB	**-3778.13**	**-3778.13**	**-3778.13**	**-3778.13**	**-3778.13**
	#L	30	time(s)	2.81	3.02	**1.63**	14.76	2.36
graph flow	#I	6	LB	-2849.96	-2853.95	-2865.20	-2888.88	**-2840.29**
	#V	$\leq$ 126	UB	-2836.57	-2829.49	-2798.11	-2181.96	**-2840.00**
	#L	$\leq$ 126	time(s)	218.22	231.53	216.39	194.43	**17.92**
car	#I	30	LB	**-69.56**	-69.57	-69.78	-69.77	-74.17
	#V	$\leq$ 49	UB	**-69.27**	-68.58	-68.32	-69.23	-57.11
	#L	$\leq$ 49	time(s)	656.86	698.74	776.58	572.41	**83.71**
motor	#I	20	LB	**-62.97**	-62.98	-63.02	-62.98	-64.25
	#V	$\leq$ 52	UB	**-62.95**	-62.83	-62.71	-62.93	-59.60
	#L	$\leq$ 52	time(s)	70.82	98.57	285.19	317.61	**69.22**
worms	#I	30	LB	**-48471.21**	-48491.81	-48517.50	-48495.92	-62934.54
	#V	$\leq$ 605	UB	**-48453.90**	-48305.90	-3316.69	-48288.65	-5254.08
	#L	$\leq$ 1500	time(s)	**213.89**	614.80	229.21	295.67	1471.95

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