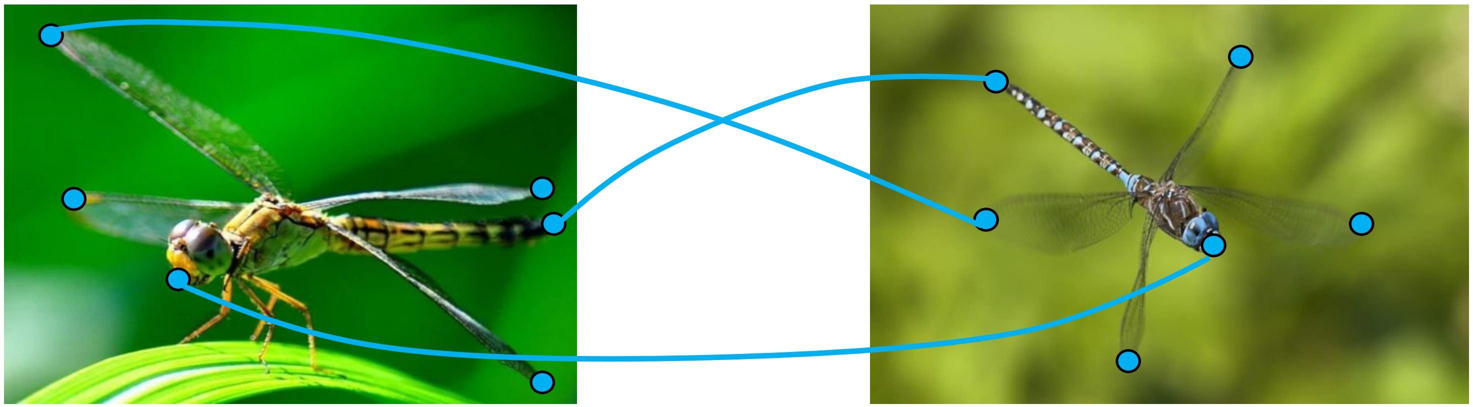
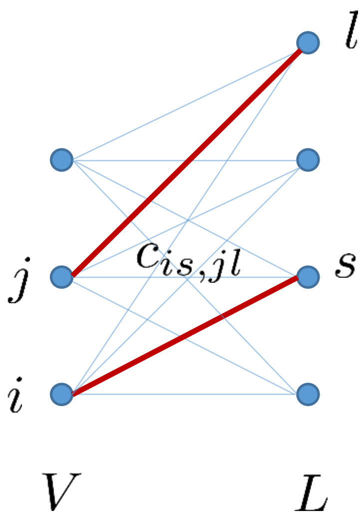
Algorithmic Techniques for Graph Matching and Their Recent Comparison Study

Bogdan Savchynskyy

24/09/2022

(Weighted) graph matching problem





$$\min_{x\in\{0,1\}^{V\times L}} \sum_{i,j\in V} \sum_{s,l\in L} c_{is,jl} x_{is} x_{jl}$$

$$\text{s.t.:} \sum_{s} x_{is} \leq 1 \;\forall i$$

$$\sum_{i} x_{is} \leq 1 \;\forall s$$

$$c_{is} = c_{is,is}$$

Related work and contribution

- More than 7 000 000 papers
- Hundreds of algorithms
- (At least) two communities: Operations research & Computer Vision
- Different goals: modeling, speed, precision

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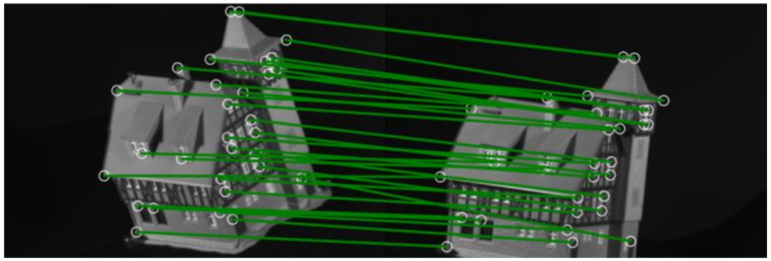
We limit our exposition to:

- Computer vision applications
- Ready cost matrix, no modeling/learning aspects
- Arbitrary costs (Lawler form)
- Open source code

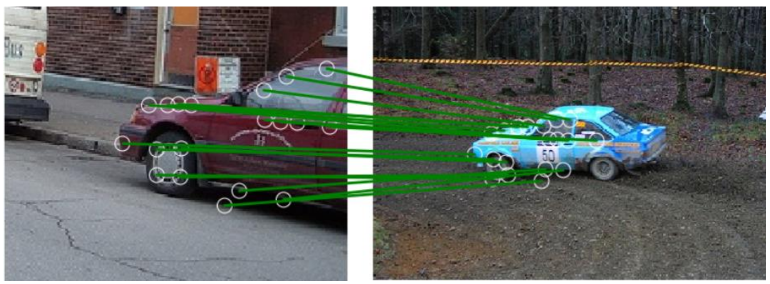
[Haller et al. 2022. A comparative study of graph matching algorithms in computer vision]

https://vislearn.github.io/gmbench/

Most popular datasets for evaluation



	hotel | **house-dense** | **house-sparse**
$\lvert V \rvert$ =	30
Density:	13% | 13% | 1.5 %
#instances:	105



	car
$\lvert V \rvert$ =	19-49
Density:	2.9%
#instances:	30

	motor(bike)
$\lvert V \rvert$ =	15-52
Density:	3.8%
#instances:	20

Most popular algorithmic baselines

- *Graduated assignment* (ga) — S. Gold, A. Rangarajan, 1996. A graduated assignment algorithm for graph matching.
- *Spectral matching* (sm) — M. Leordeanu, M. Hebert, 2005. A spectral technique for correspondence problems using pairwise constraints.
- *Spectral matching with affine constraints* (smac) — T. Cour, P. Srinivasan, J. Shi, 2007. Balanced graph matching
- *Integer projected fixed point* (ipfp-u/s) — M. Leordeanu, M. Hebert, R. Sukthankar, 2009. An integer projected fixed point method for graph matching and MAP inference
- *Reweighted random walks matching* (rrwm) — M. Cho, J. Lee, K. Mu Lee, 2010. Reweighted random walks for graph matching
- *Local sparse model* (lsm) — B. Jiang, J. Tang, C. Ding, B. Luo, 2015. A local sparse model for matching problem
- *Factorized graph matching* (ipfp-u/s) — F. Zhou, F. de la Torre, 2016. Factorized Graph Matching

Our findings

- Most popular datasets are easy to solve (exactly, < 1 sec)
- Most popular algorithmic baselines are very weak
- Even large problems can be solved fast:

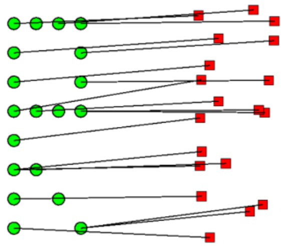
  V > 500, L > 1300, t < 1 sec

Outline

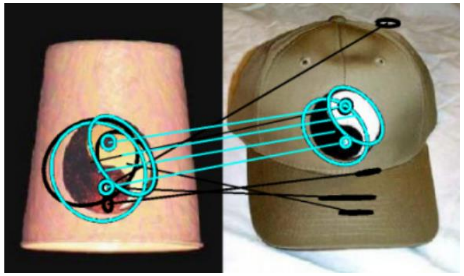
- Empirical evaluation
- Design of the best performing methods

Empirical evaluation

What additional datasets are in our benchmark



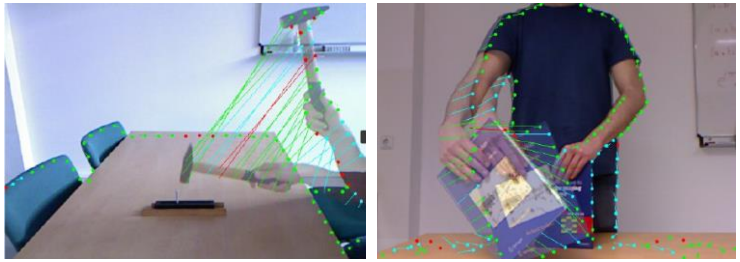
opengm
$|V|$ = 19-20
Density: 75%
#instances: 4



caltech-small
$|V|$ = 9-117
Density: 1%
#instances: 21

Additional small instances: $|V| < 120$

What datasets are in our benchmark

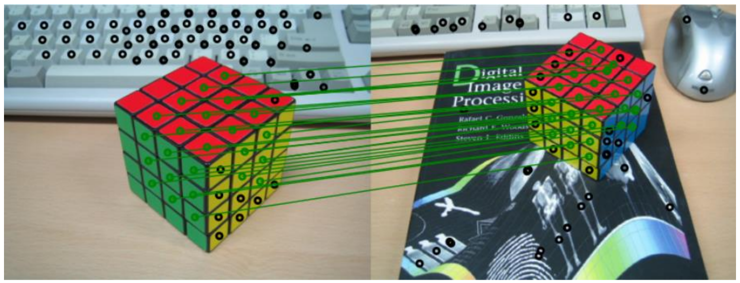


flow

$|V|$ = 48-126

Density: 0.4%

#instances: 6

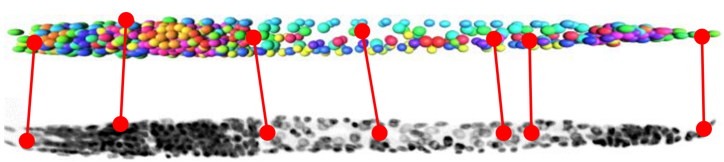


caltech-large

$|V|$ = 36-219

Density: 0.55%

#instances: 9

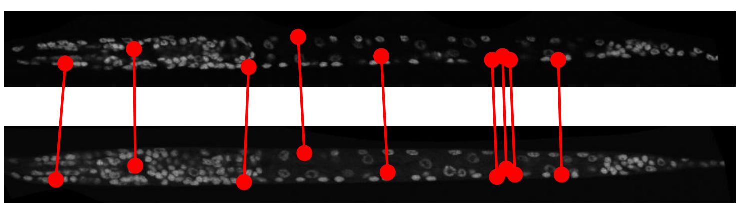


worms (atlas)

$|V|/|L|$ = 558/1300

Density: 0.00038%

#instances: 30



(worm) pairs

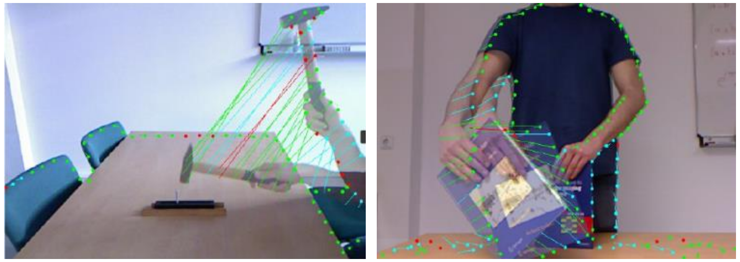
$|V|$ = 511-565

Density: 0.0019%

#instances: 16

Additional large instances: $|V| > 120$

What datasets are in our benchmark



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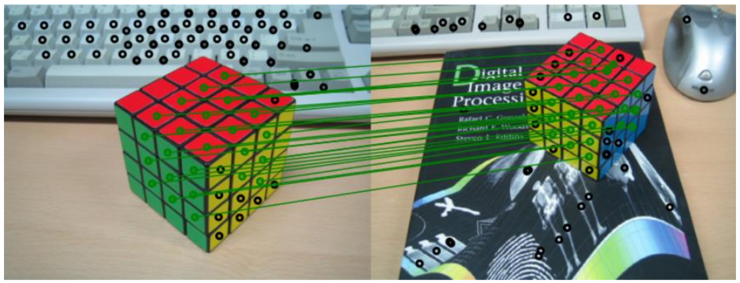

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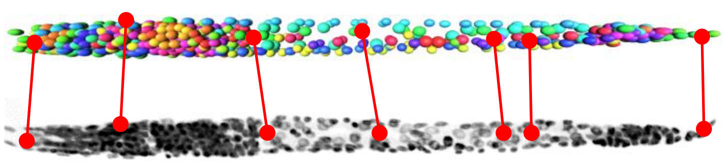

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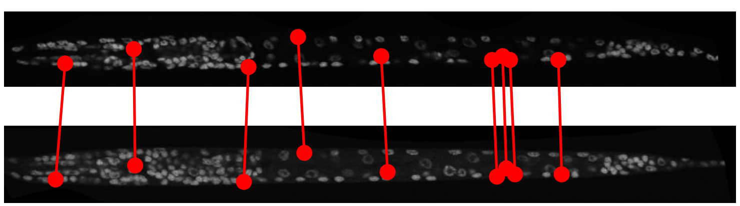


(worm) pairs

$|V|$ = 511-565
Density: 0.0019%
#instances: 16

Additional large instances: $|V| > 120$

11 datasets, 451 problem instances in total

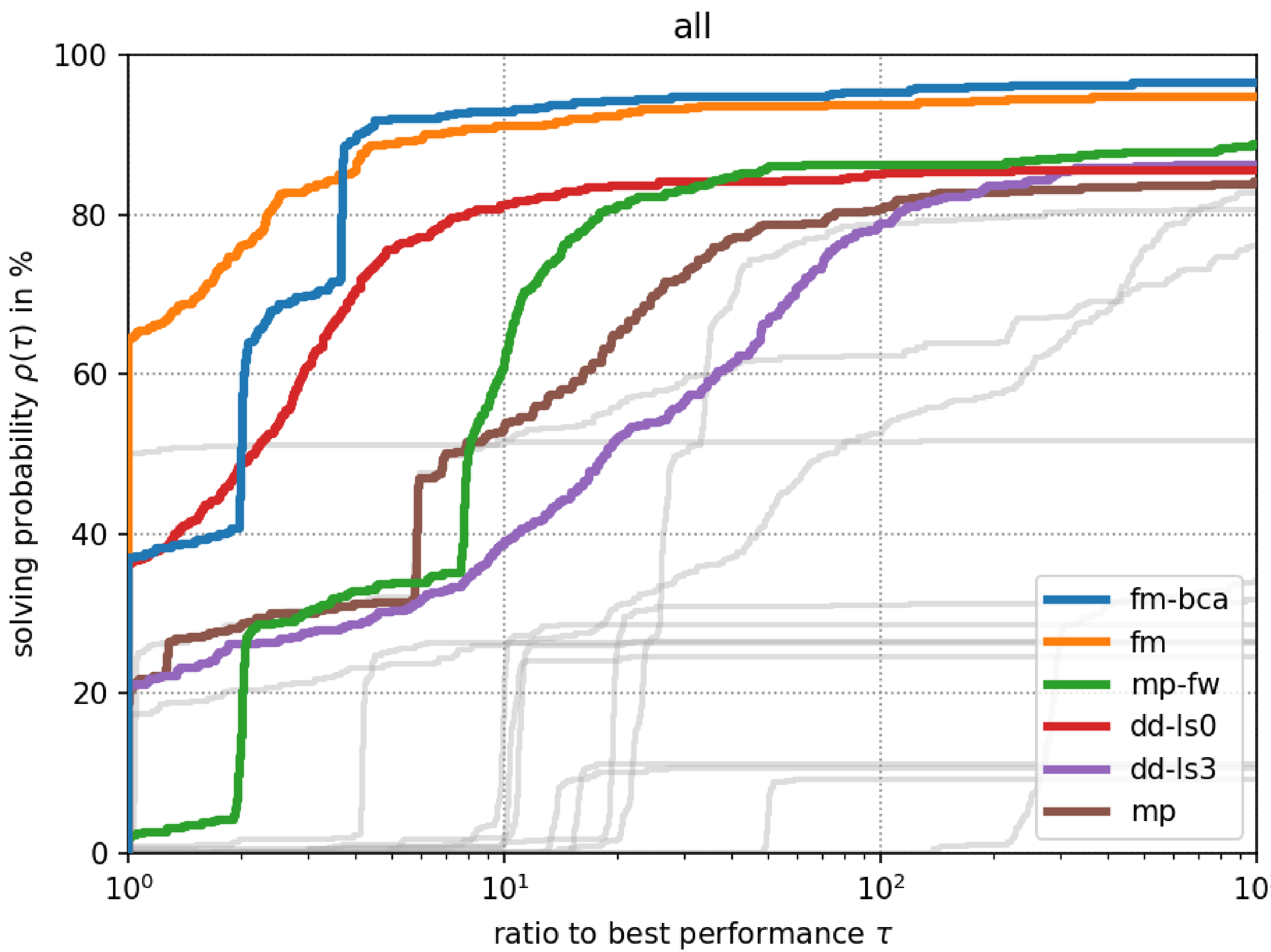
What algorithms we evaluate

	method	IQP	ILP	bijective	non-pos.	0-unary	lineariz.	norm	doubly	spectral	discret.	path fol.	fusion	duality	SGA	BCA	Matlab	C++
Primal methods	fgmd [69]	+		+								+					[68]	
	fm [31]		+										+					[32]
	fw [62]	+					+		+									[56]
	ga [27]	+		+			+		+		+						[20]	
	ipfps [46]	+		+	+		+		+		+						[44]	
	ipfpu [46]	+		+			+		+		+						[44]	
	lsm [33]	+		+	+			+			+						[66]	
	mpm [18]	+		+	+			+			+						[19]	
	pm [65]	+		+	+	+			+		+						[68]	
	rrwm [17]	+		+			+		+		+						[16]	
	smac [21]	+		+	+			+		+	+						[20]	
	sm [43]	+		+	+			+		+	+						[44]	
Duality-based	dd-ls(0/3/4) [59]		+											+	+			[38]
	fm-bca [31]		+										+	+		+		[32]
	hbp [67]		+	+										+		+	[66]	
	mp(-mcf/-fw) [57]		+											+		+		[56]

Most popular baselines

20 algorithms

What are the results of evaluation



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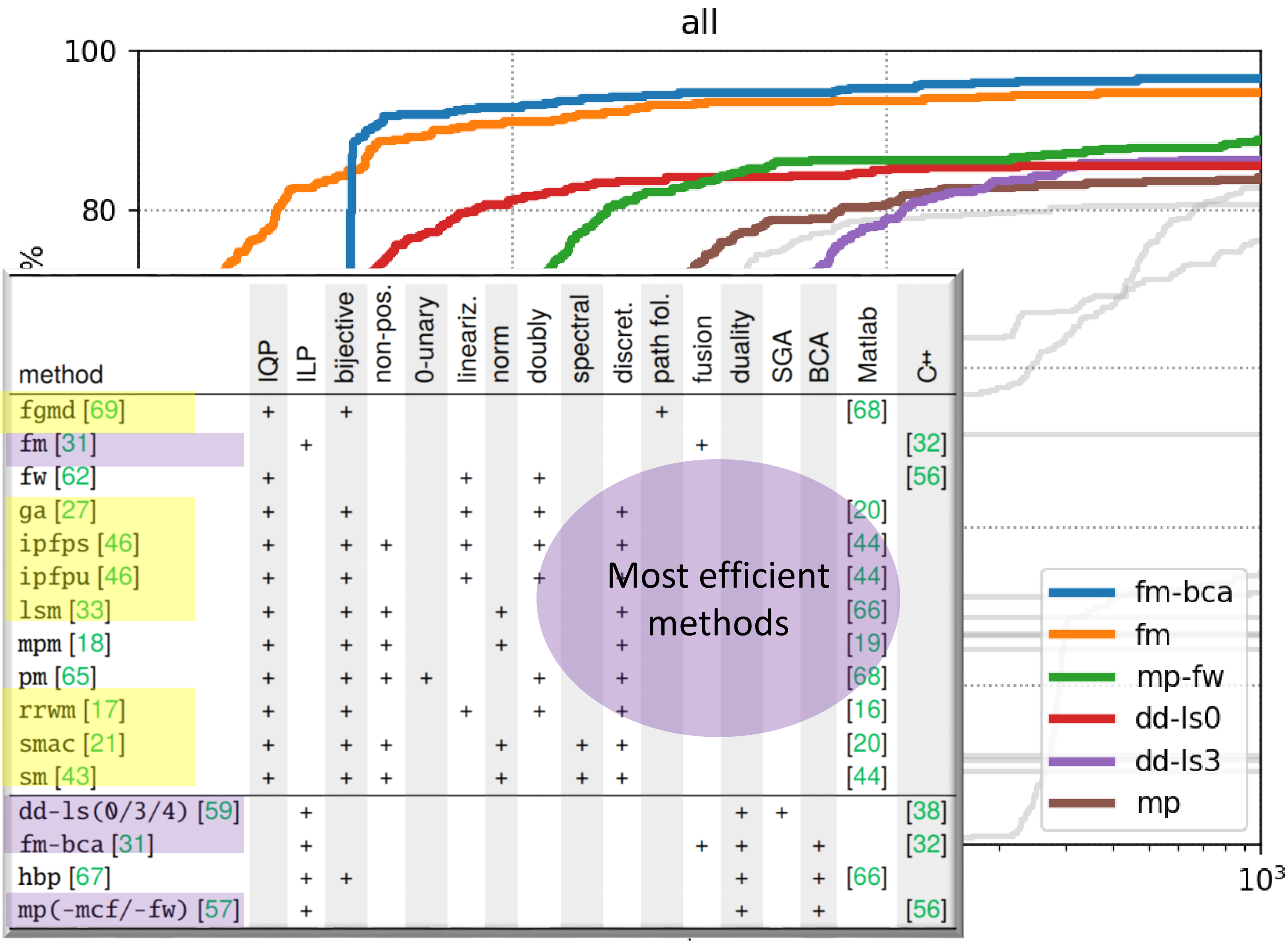

Primal methods / Duality-based

method	IQP	ILP	bijective	non-pos.	0-unary	lineariz.	norm	doubly	spectral	discret.	path fol.	fusion	duality	SGA	BCA	Matlab	C++
fgmd [69]	+		+								+					[68]	
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fw [62]	+					+		+									[56]
ga [27]	+		+			+		+		+						[20]	
ipfps [46]	+		+	+		+		+		+						[44]	
ipfpu [46]	+		+			+		+								[44]	
lsm [33]	+		+	+			+			+						[66]	
mpm [18]	+		+	+			+			+						[19]	
pm [65]	+		+	+	+			+		+						[68]	
rrwm [17]	+		+			+		+		+						[16]	
smac [21]	+		+	+			+		+	+						[20]	
sm [43]	+		+	+			+		+	+						[44]	
dd-ls(0/3/4) [59]		+											+	+			[38]
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hbp [67]		+	+										+		+	[66]	
mp(-mcf/-fw) [57]		+											+		+		[56]

Most efficient methods

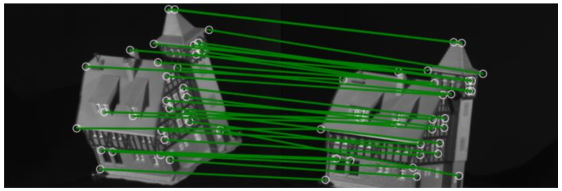
Evaluation peculiarities

- t = 1, 10, 100, 300 sec.
- Methods iteratively output current results.

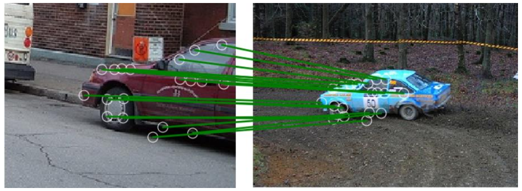
- Evaluation criteria: # optima (opt), *E* (obj), dual bound, accuracy (acc)

- Optimal solutions obtained with Gurobi or *E*=dual bound:
- Optimum found: 416 instances
- Optimum unknown: 35 instances

Most popular datasets



hotel | house-dense | house-sparse



cars

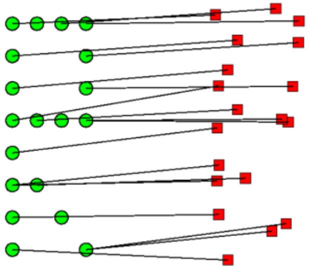
motor

	hotel		house-dense		house-sparse		cars		motor	
	opt %	**acc %**	**opt %**	**acc %**	**opt %**	**acc %**	**opt %**	**acc %**	**opt %**	**acc %**
fm-bca	**100**	**100**	**100**	**100**	**100**	**100**	93	**92**	**100**	**97**
dd-ls0	**100**	**100**	**100**	**100**	**100**	**100**	**97**	91	**100**	**97**
fgmd	96	98	77	89	**100**	**100**	83	89	85	**97**

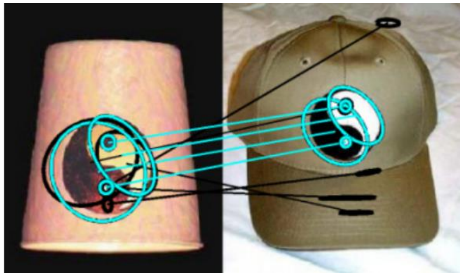
Max. run-time: 1 sec, fgmd – 300 sec

All solved in < 1 sec!

Small datasets



opengm

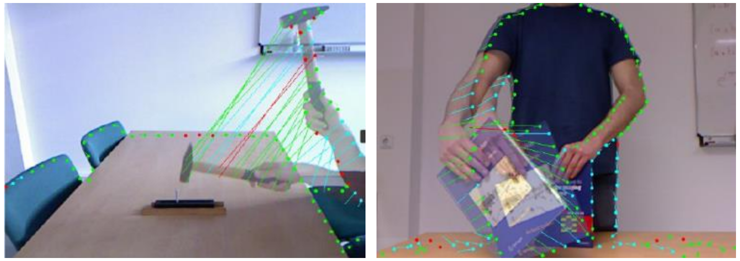


caltech-small

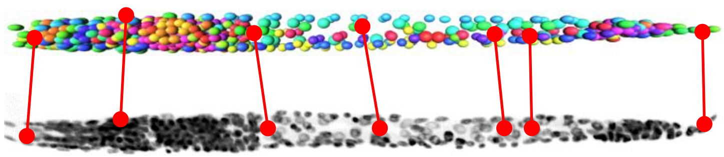
	opengm		caltech-small		
	opt %	**obj**	**opt %**	**obj**	**acc %**
fm	**100**	**-171**	**57**	**-9040**	62
fm-bca	**100**	**-171**	43	-8943	62
fgmd	75	-166	43	-	-
ipfps	0	-95	19	-8983	**67**

Max. run-time: 10 sec, fgmd – 300 sec

Large datasets



flow



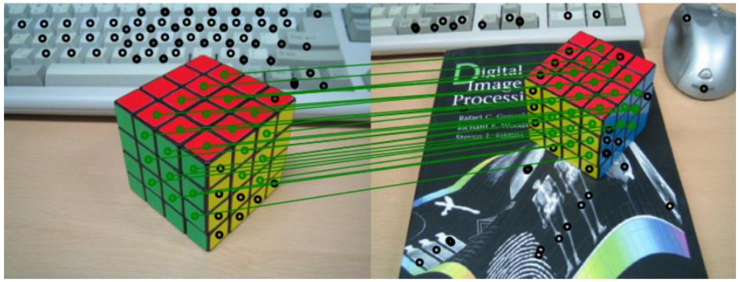
Worms, $|V|/|L|$ = 558/1300

	flow		worms	
	opt %	**obj**	**opt %**	**acc %**
fm	**100**	**-2840**	**93**	**89**
fm-bca	**100**	**-2840**	**93**	**89**
ga	0	-2469	0	0

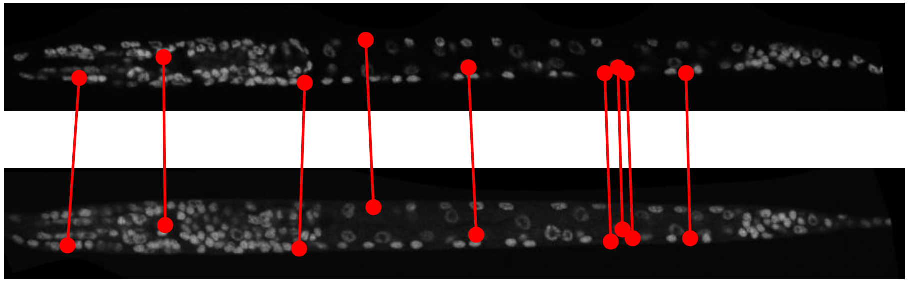
Max. run-time: 10 sec, ga – 300 sec

33 out of 36 instances optimally solved in < 1 sec!

Large datasets



caltech-large



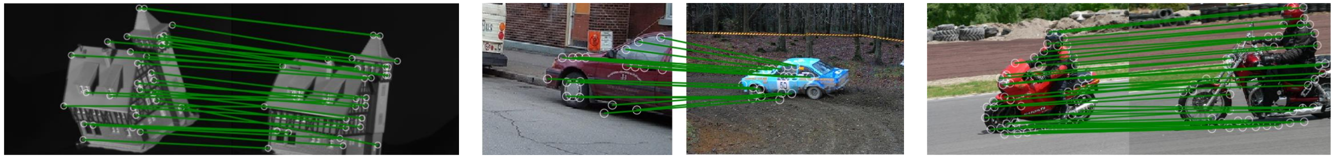
pairs

	caltech-large		pairs	
	obj	**acc %**	**obj**	**acc %**
fm-bca	-34039	51	**-65913**	**58**
fm	**-34117**	51.6	-65625	**54**
ipfpu	**-34216**	**52**	-35666	7

Max. run-time: 10 sec

Empirical conclusions

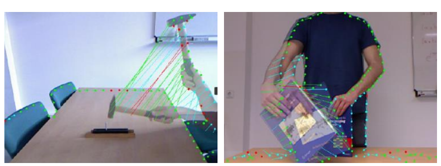
- Popular datasets are insufficient to show efficiency of new algorithms

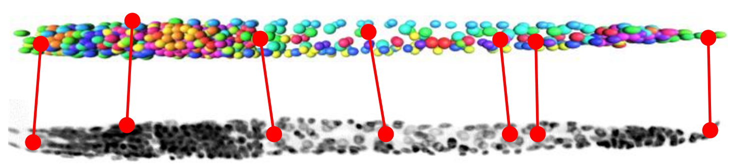


- Popular algorithms are insufficient as baselines

	flow		worms	
	opt %	**obj**	**opt %**	**acc %**
ga	0	-2469	0	0

- The most efficient methods are duality-based ones (e.g. fm-bca,dd-ls0,mp-fw) equipped with powerful primal heuristics
- Even large problem instances with |V|>200-500 can often be solved in < 1 sec





Ready to deep-learn in large scale?

Design of the best performing methods

Design of the best performing methods

1. Based on integer linear program (**ILP**) **representation**
2. **Optimize Lagrange dual** and simplify the cost structure of the problem
3. Run a **strong primal heuristic** on the simplified costs

- *Fusion moves* (fm, fm-bca) — L. Hutschenreiter et al. 2021, Fusion moves for graph matching
- *Message passing* (mp-XXX) — P. Swoboda et al. 2017, A study of Lagrangean decompositions and dual ascent solvers for graph matching
- *Dual decomposition*(dd-lsX) — L. Torresani et al. 2013, A dual decomposition approach to feature correspondence

Strong primal heuristic

3. Strong primal heuristic: Fusion moves

1. Generate proposal solutions
2. Fuse/merge/recombine/crossover it with the current one
3. Update the current solution
4. Goto 1.

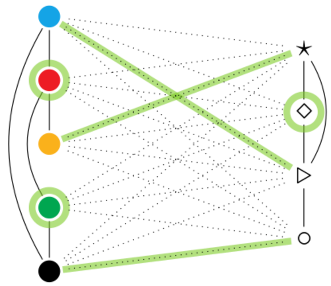
- ❑ *Fusion moves* (fm, fm-bca)

L. Hutschenreiter et al. 2021, Fusion moves for graph matching

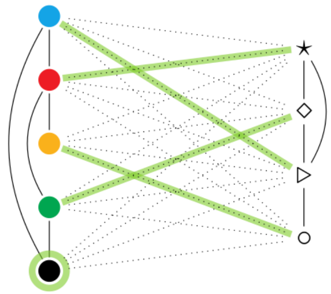
Fusion operation

3. Strong primal heuristic: 2) Fusion operation

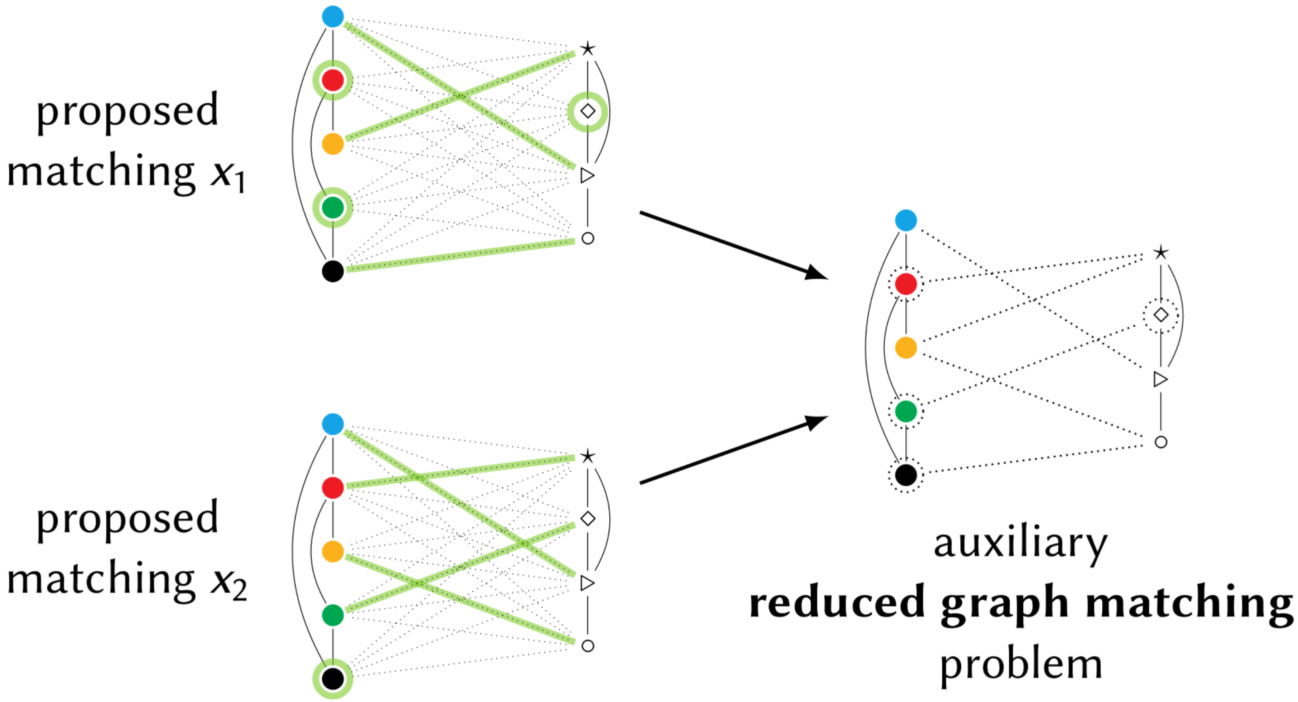
proposed matching x_1



proposed matching x_2



3. Strong primal heuristic: 2) Fusion operation



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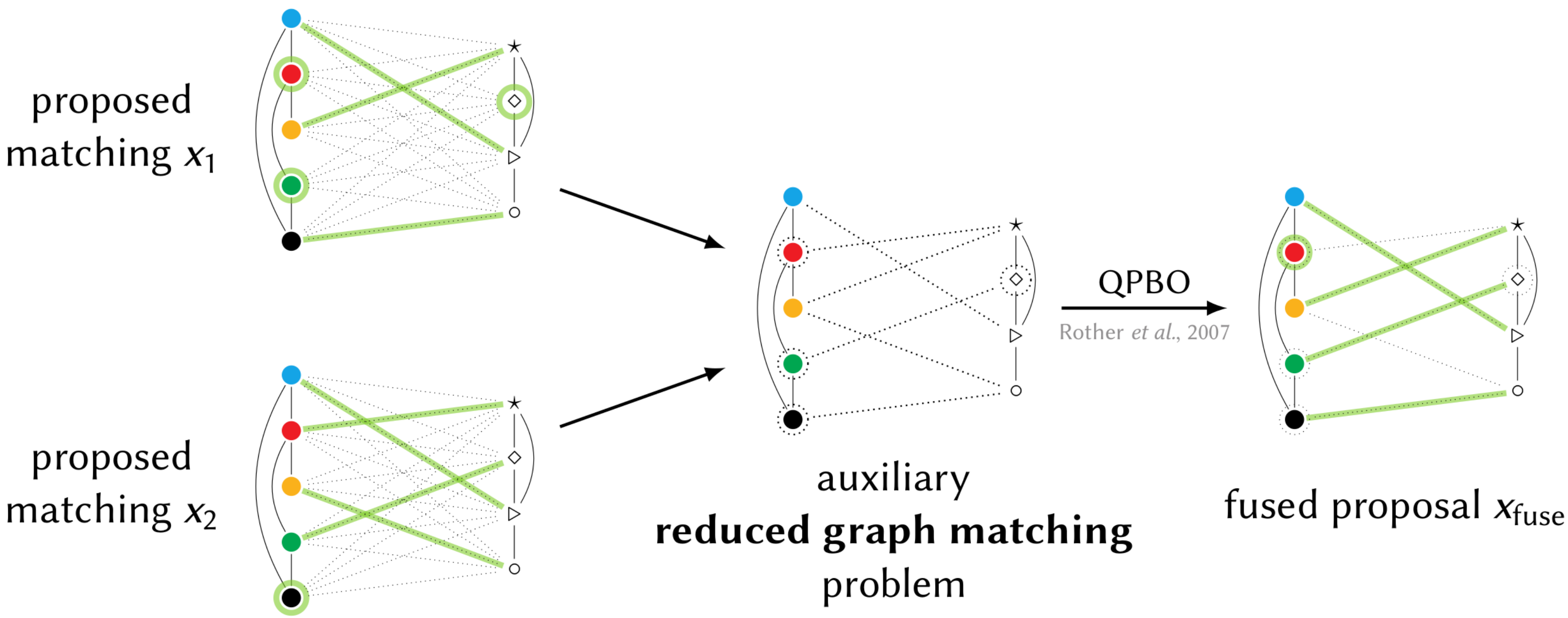


$$E(\theta, x_{\text{fuse}}) \leq \min\left[E(\theta, x_1), E(\theta, x_2)\right]$$

Proposal generation

Proposal generation

Desired properties:

- **Diverse**
- **Low objective value**

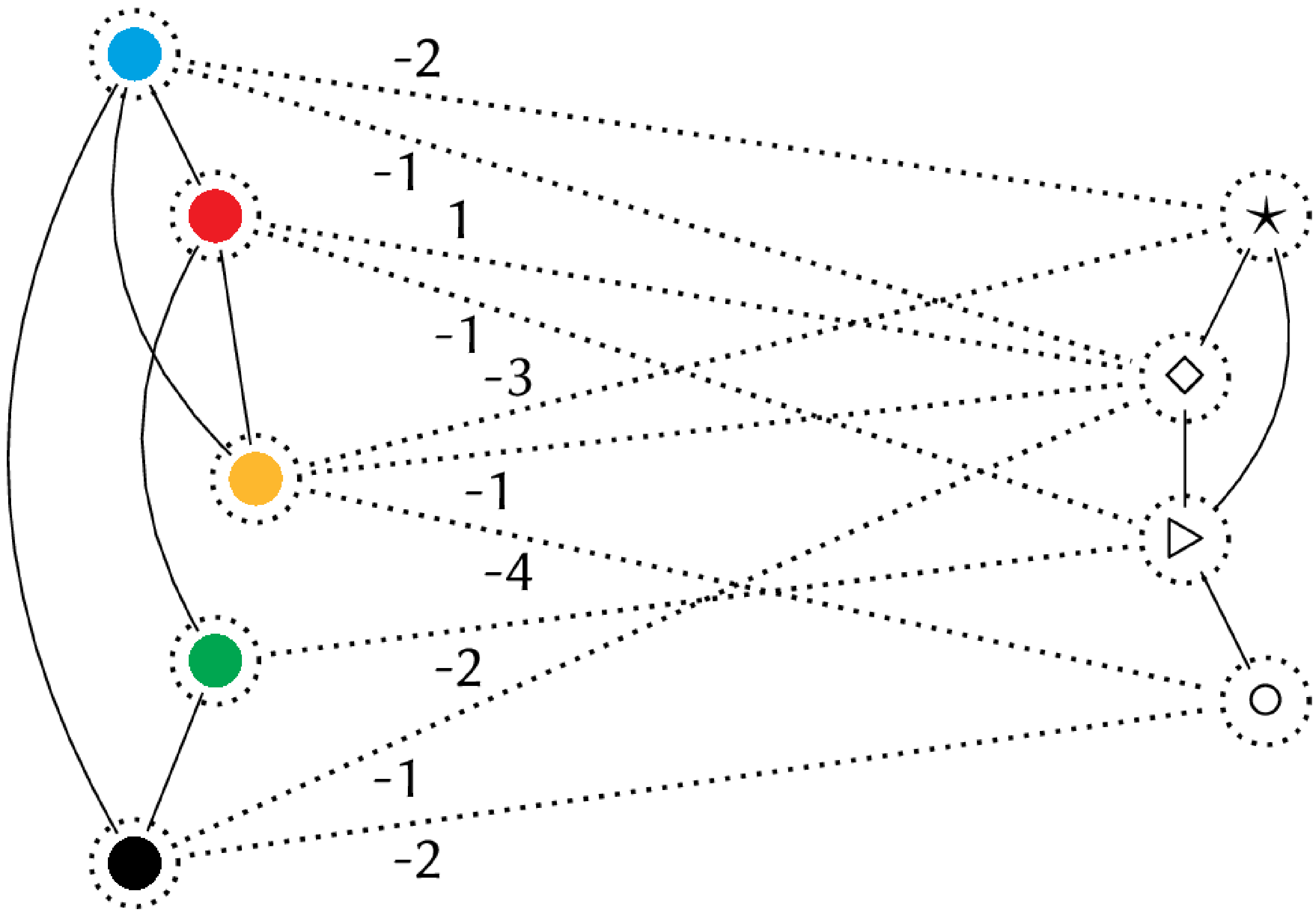
Proposal generation

Desired properties:

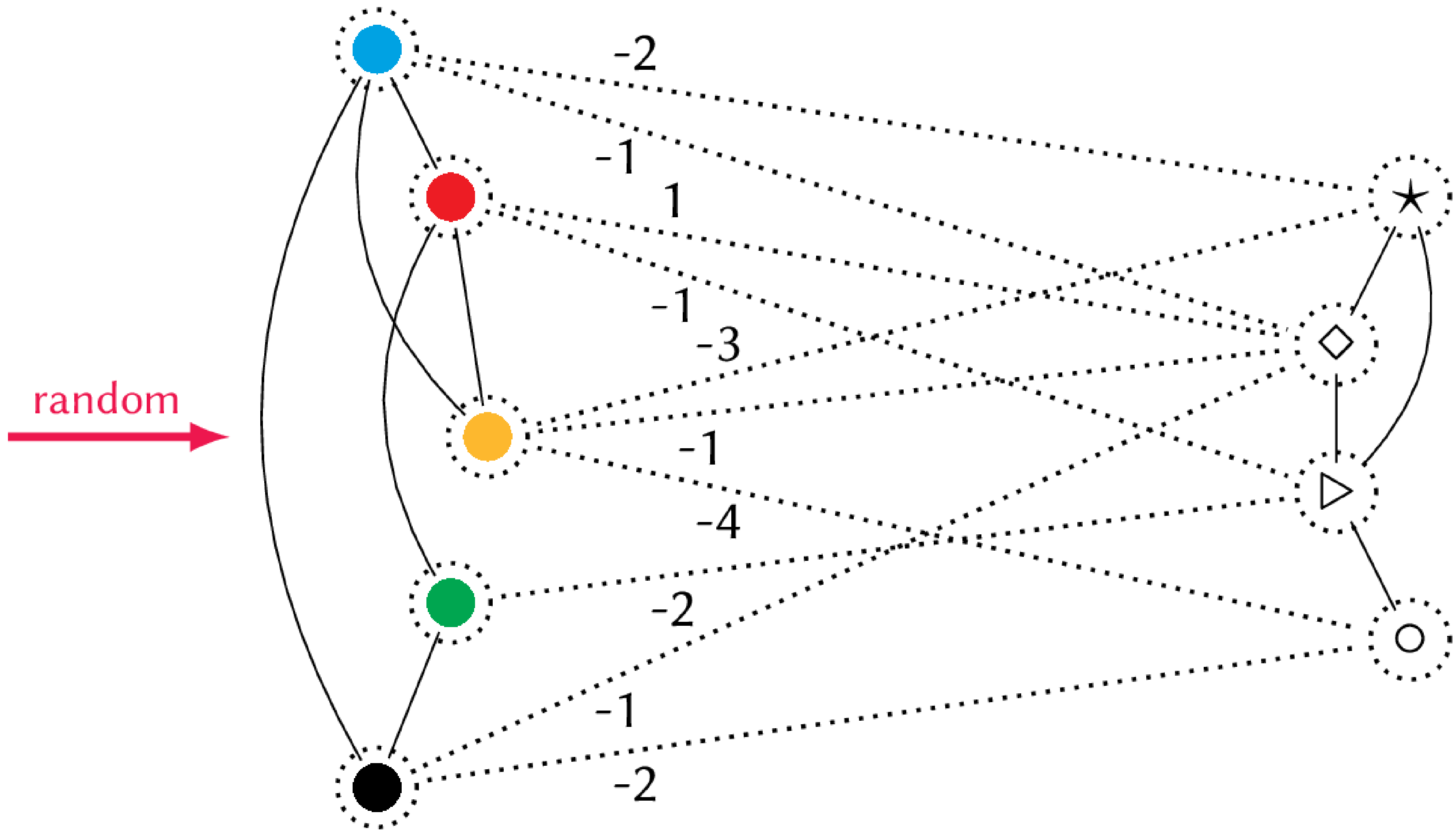
- **Diverse** →
- **Low objective value** →

Randomized greedy heuristic

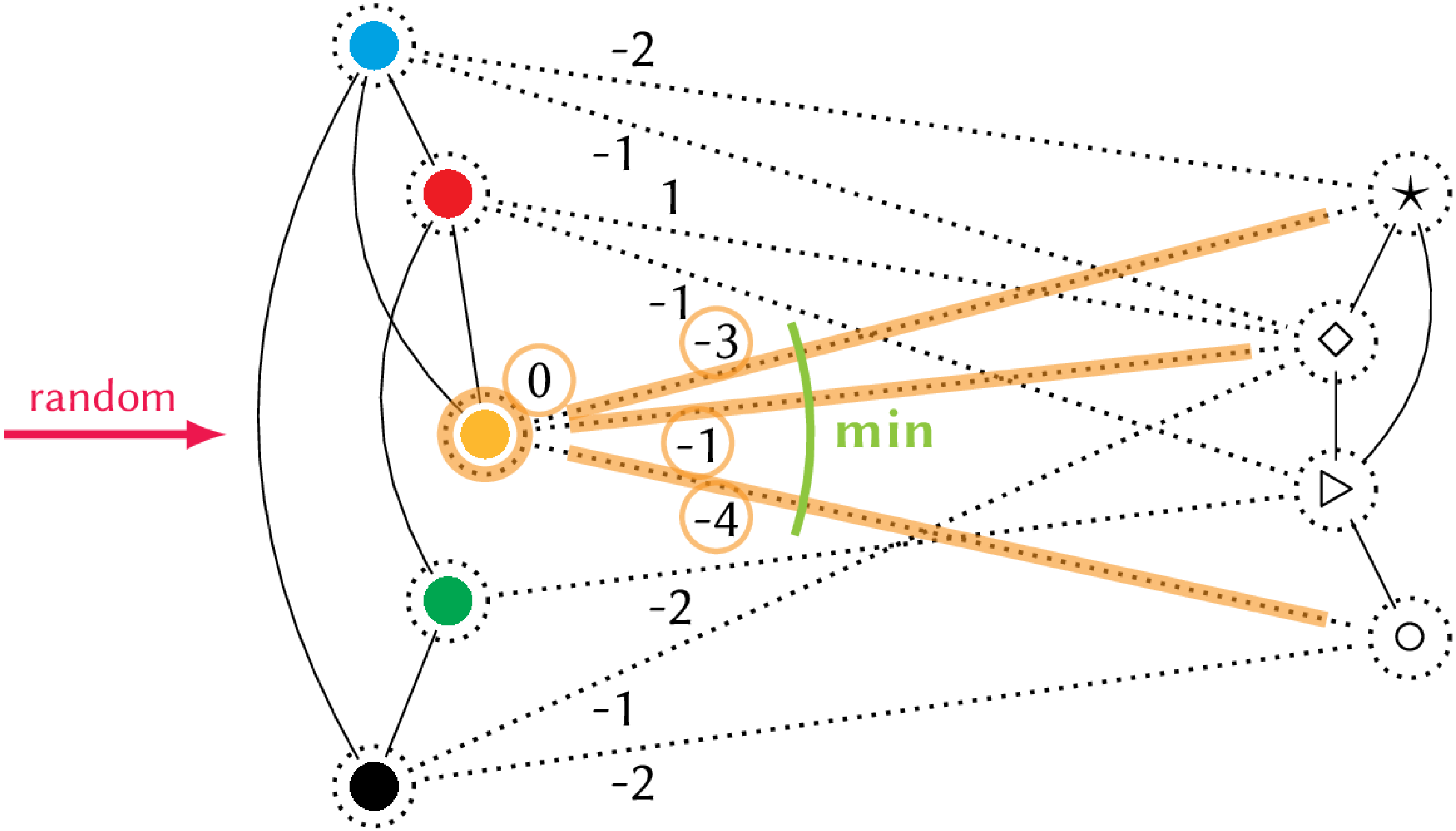
Randomized greedy heuristic



Randomized greedy heuristic

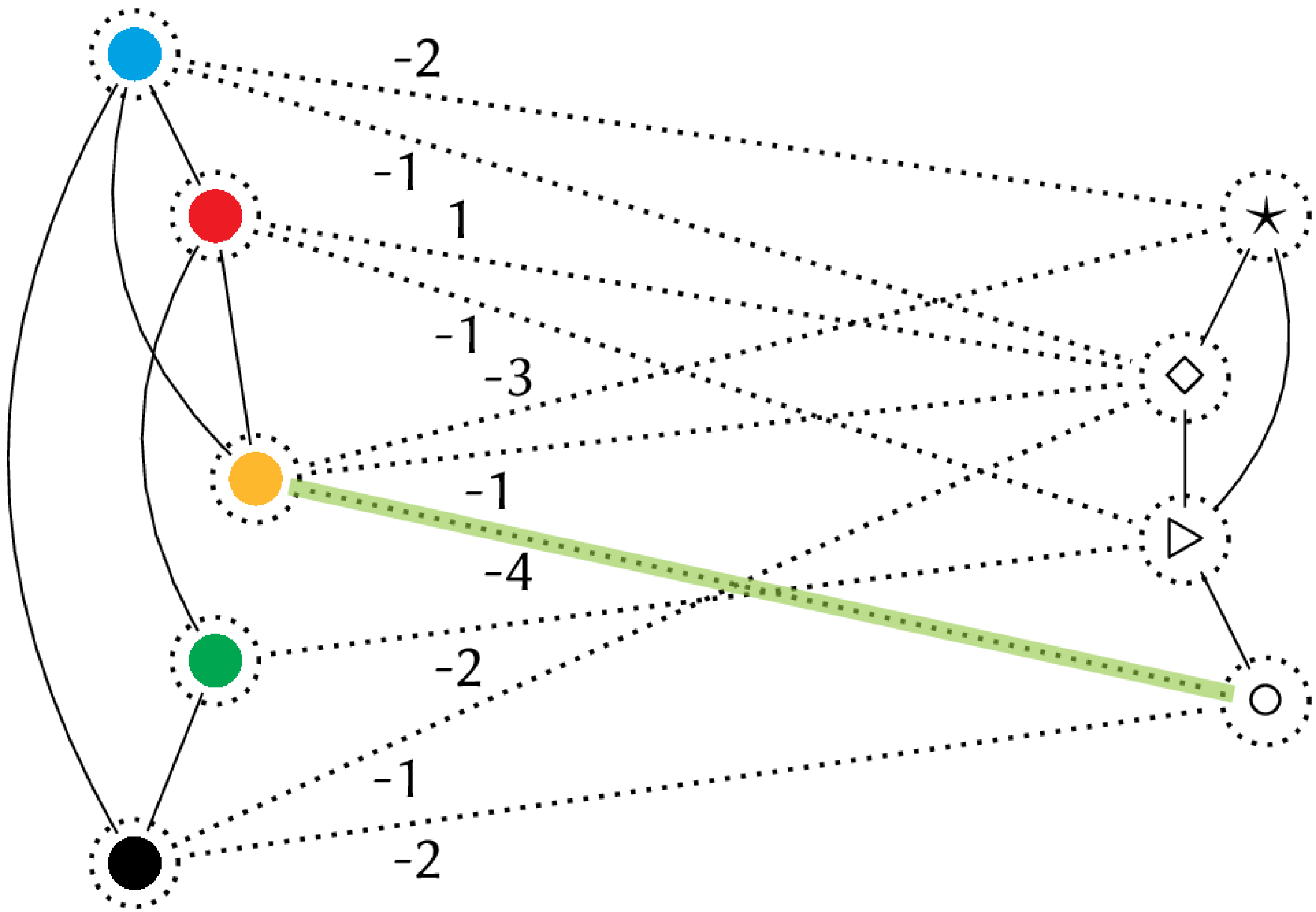


Randomized greedy heuristic

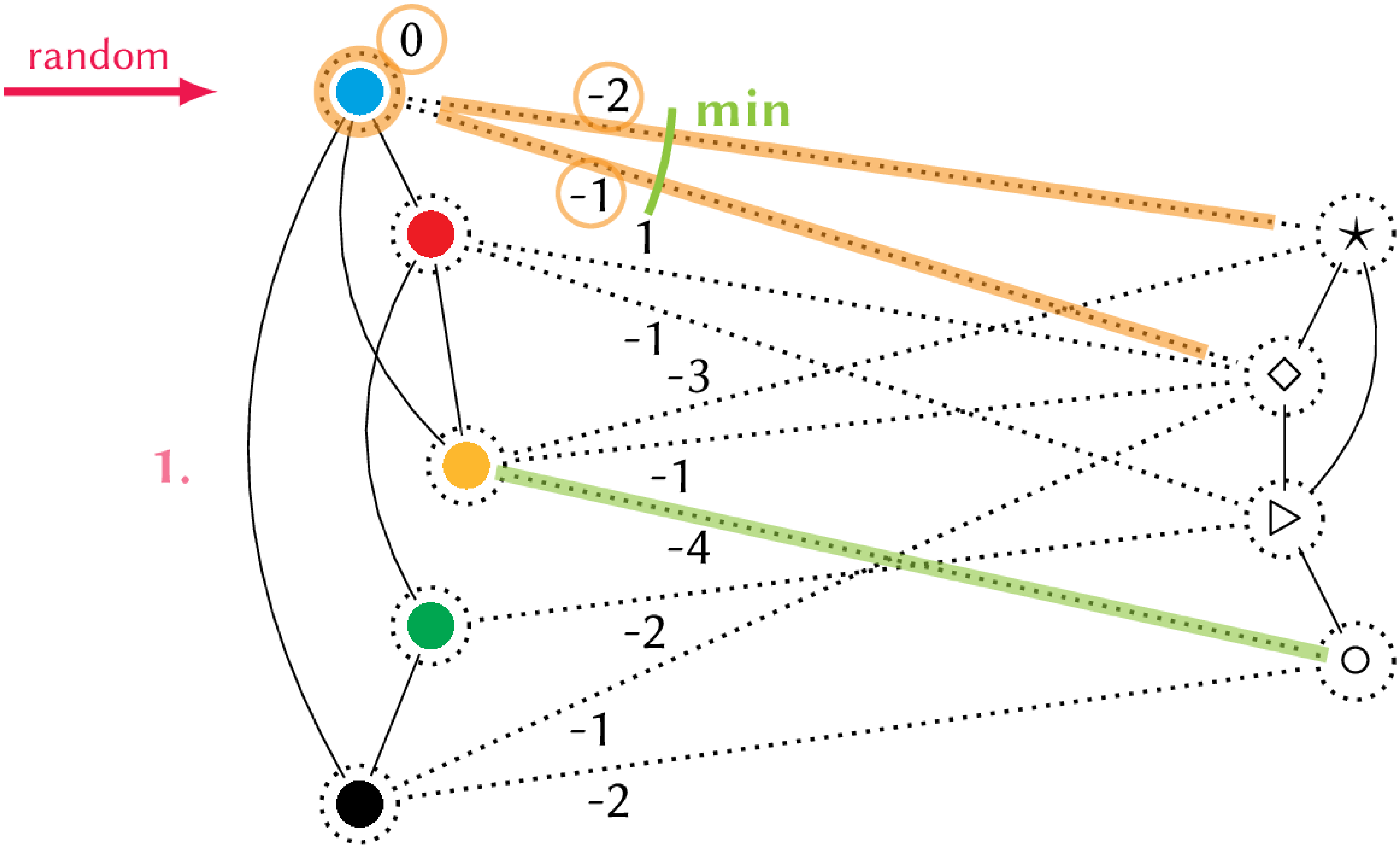


Randomized greedy heuristic

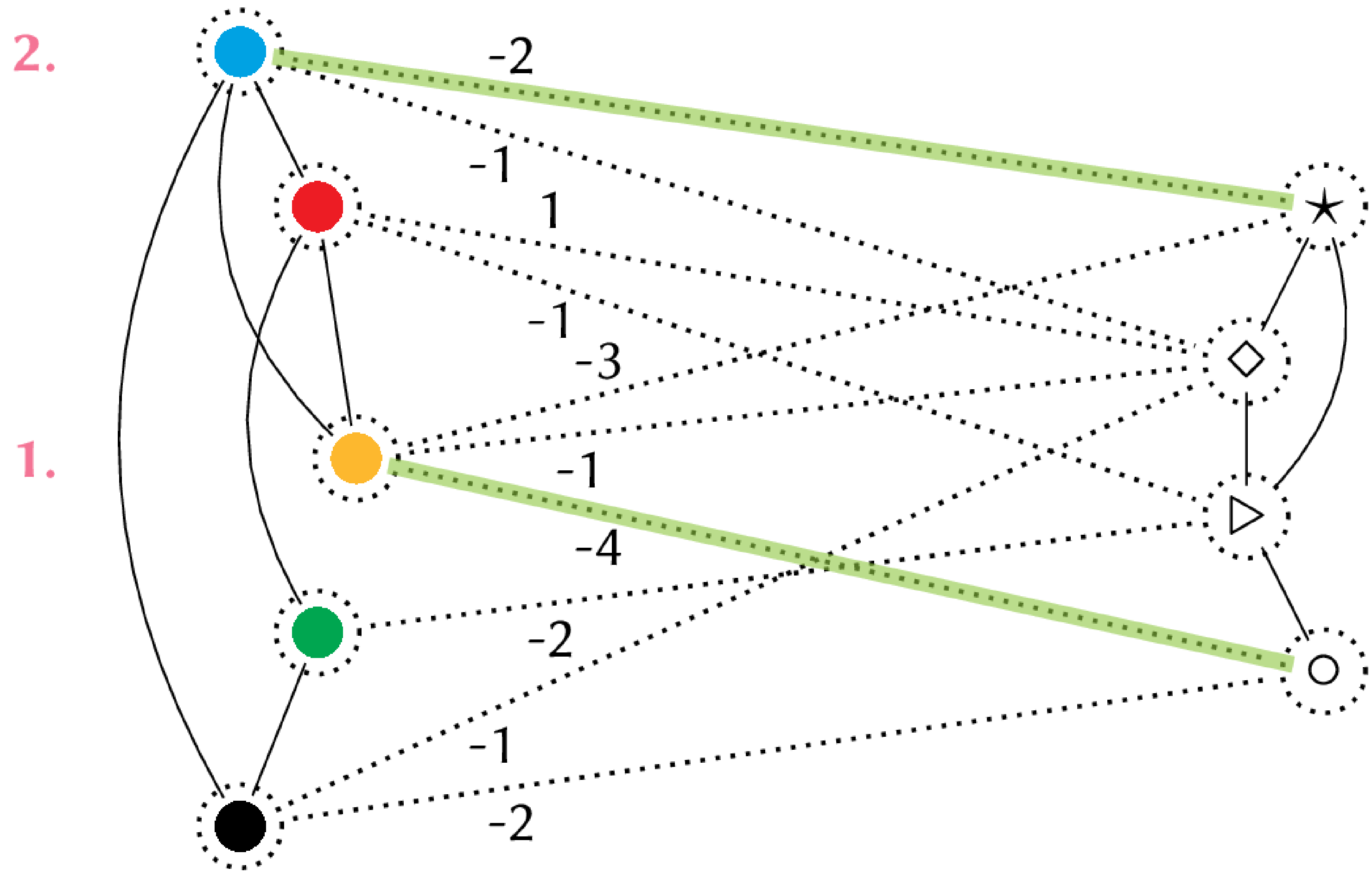
1.



Randomized greedy heuristic



Randomized greedy heuristic



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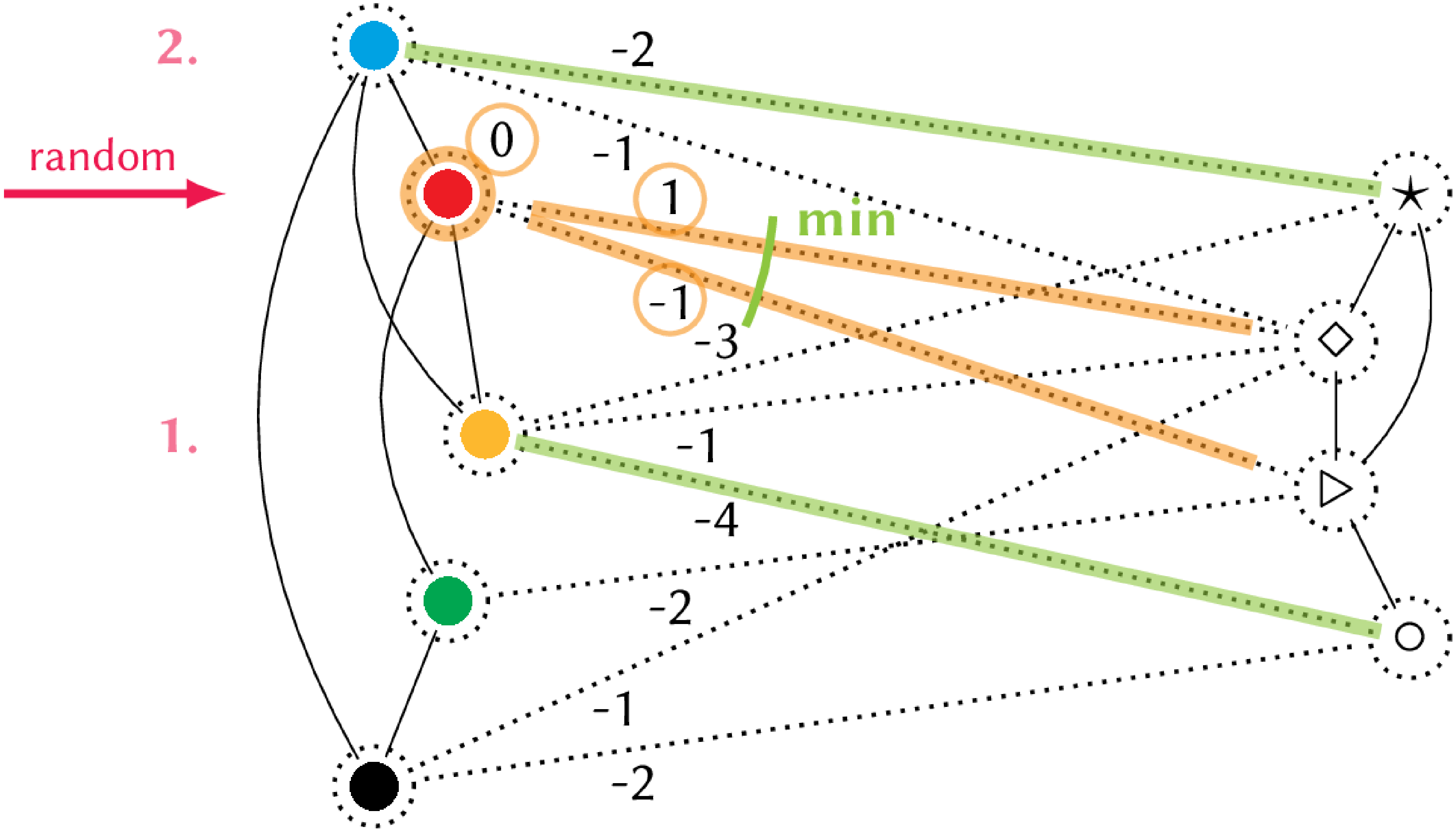

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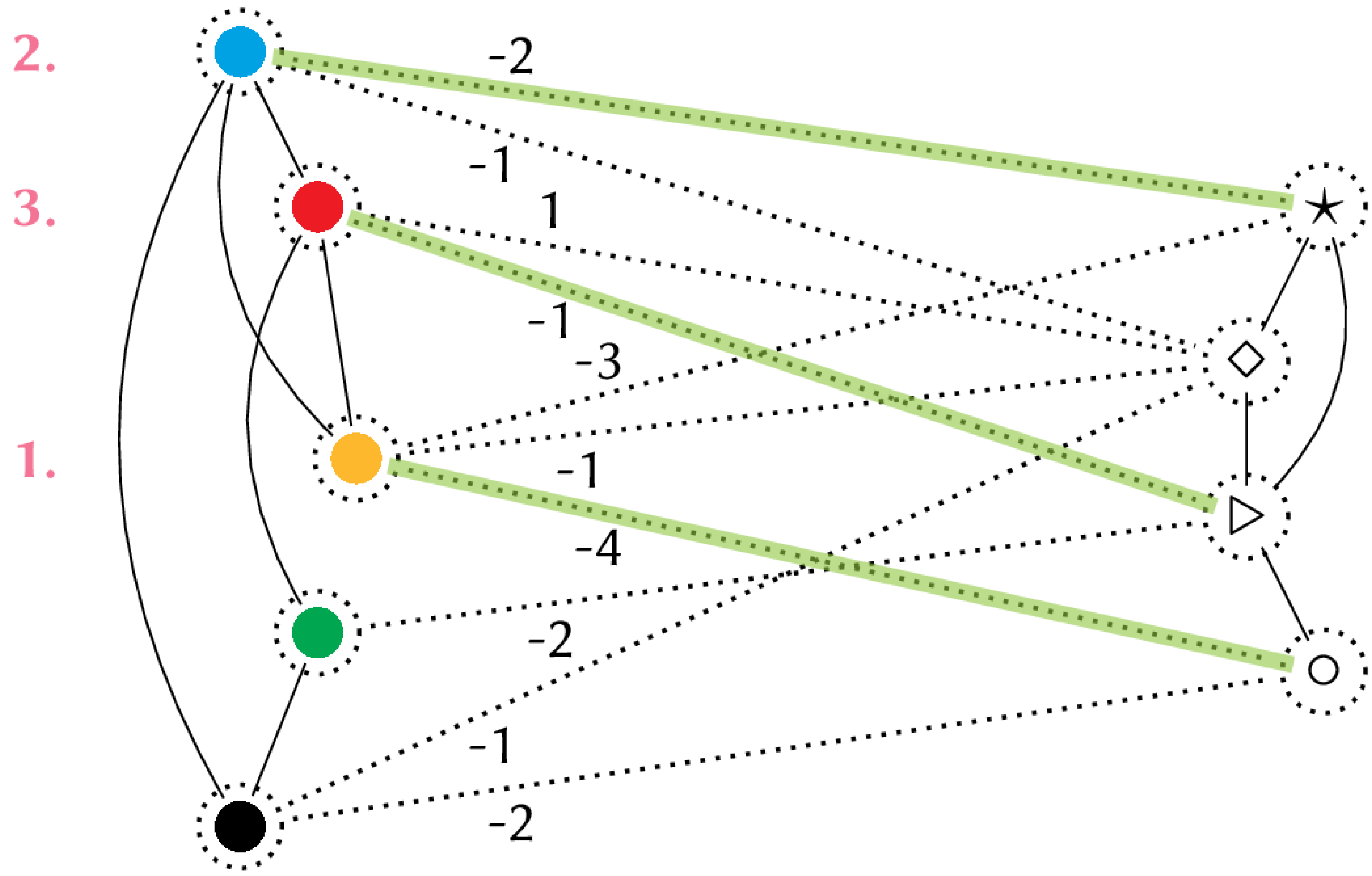

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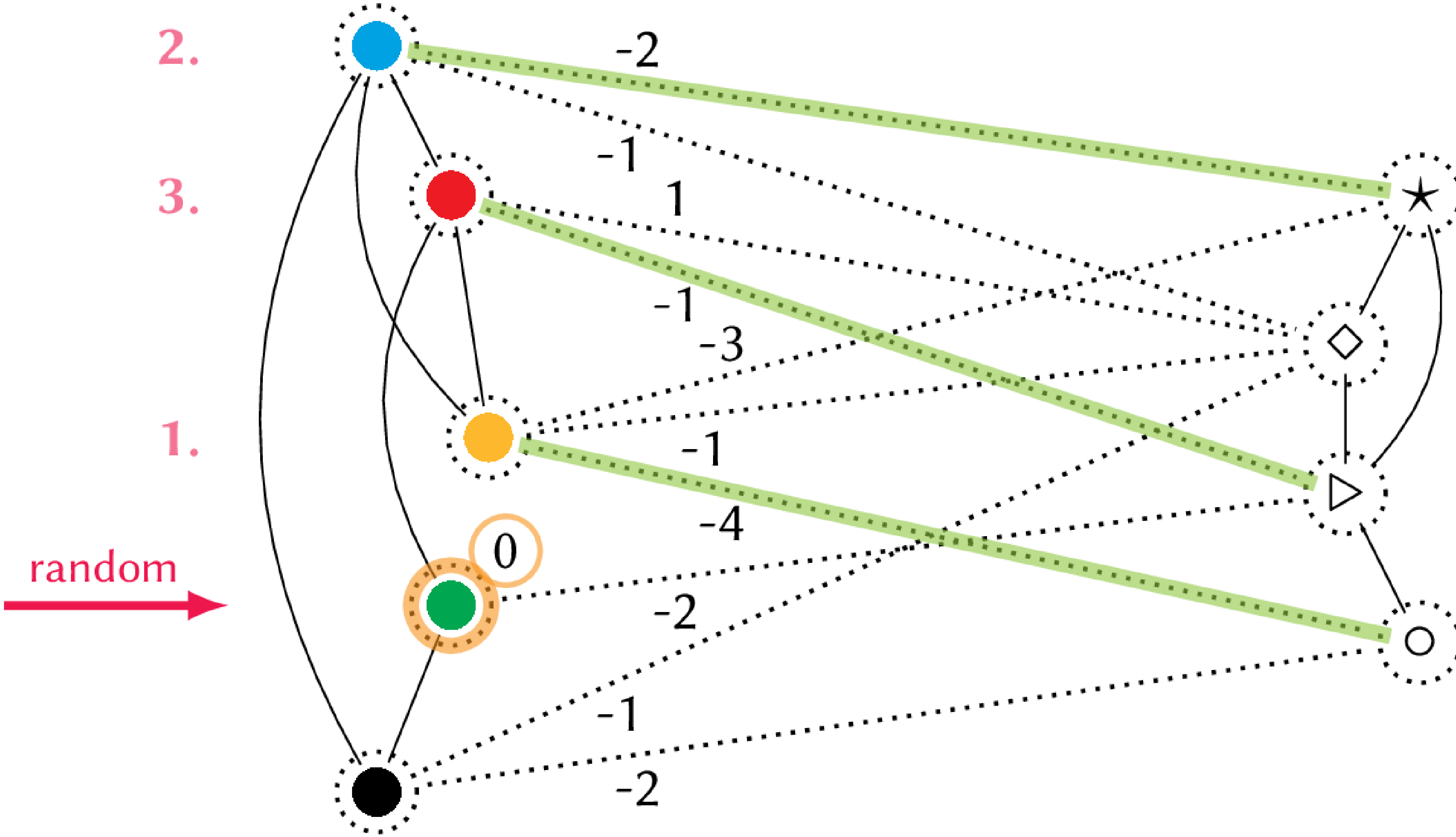

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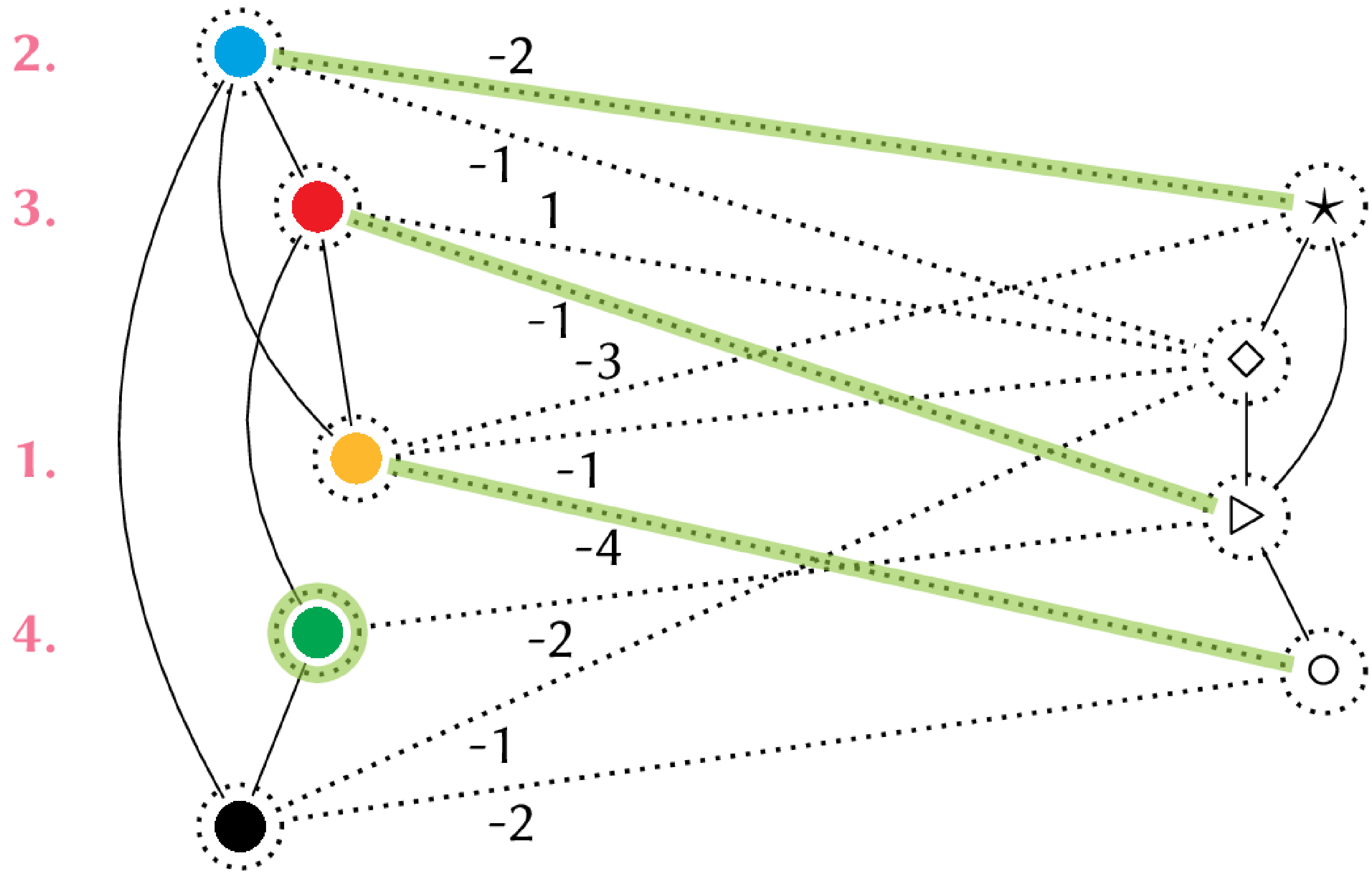

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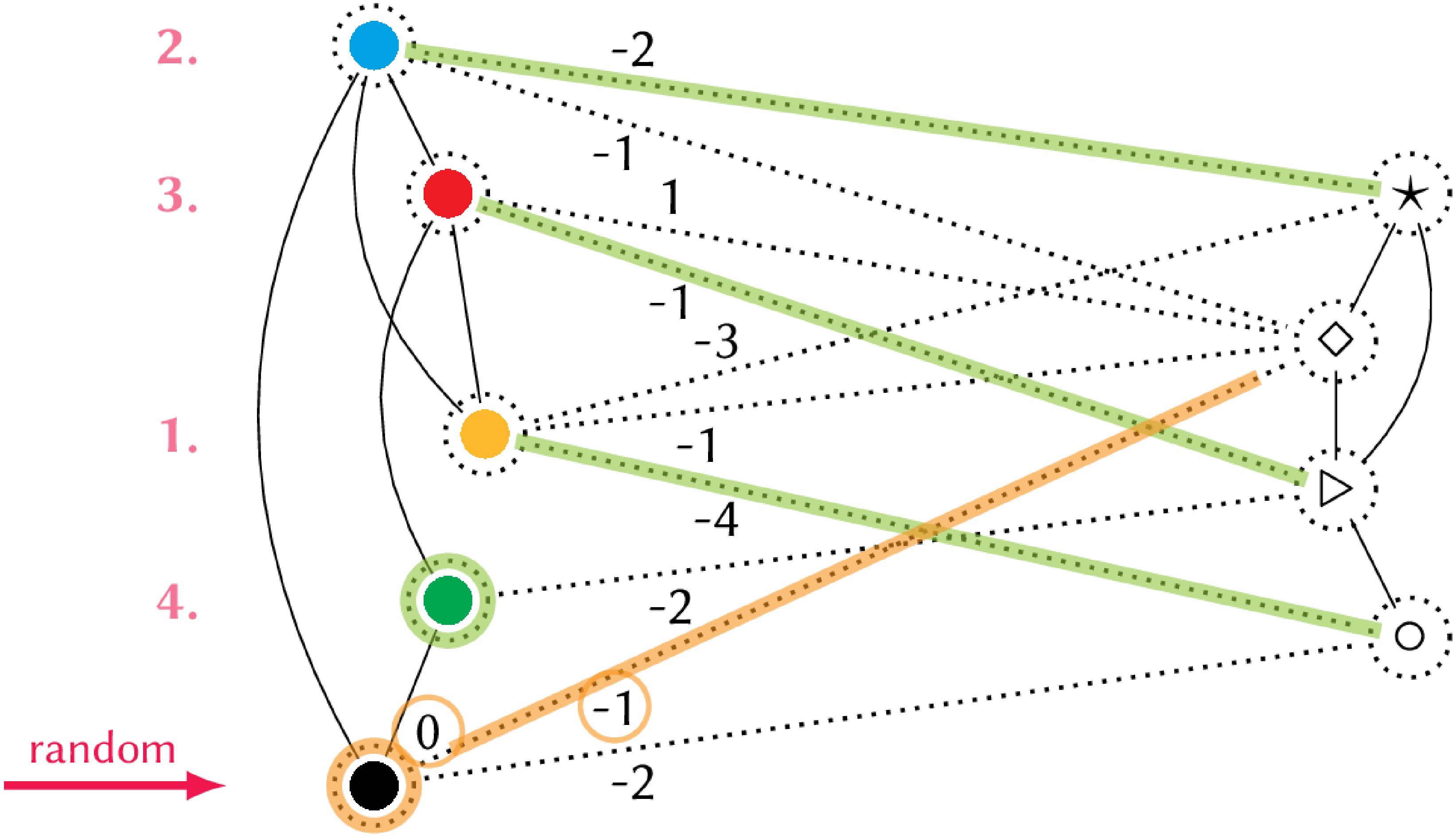

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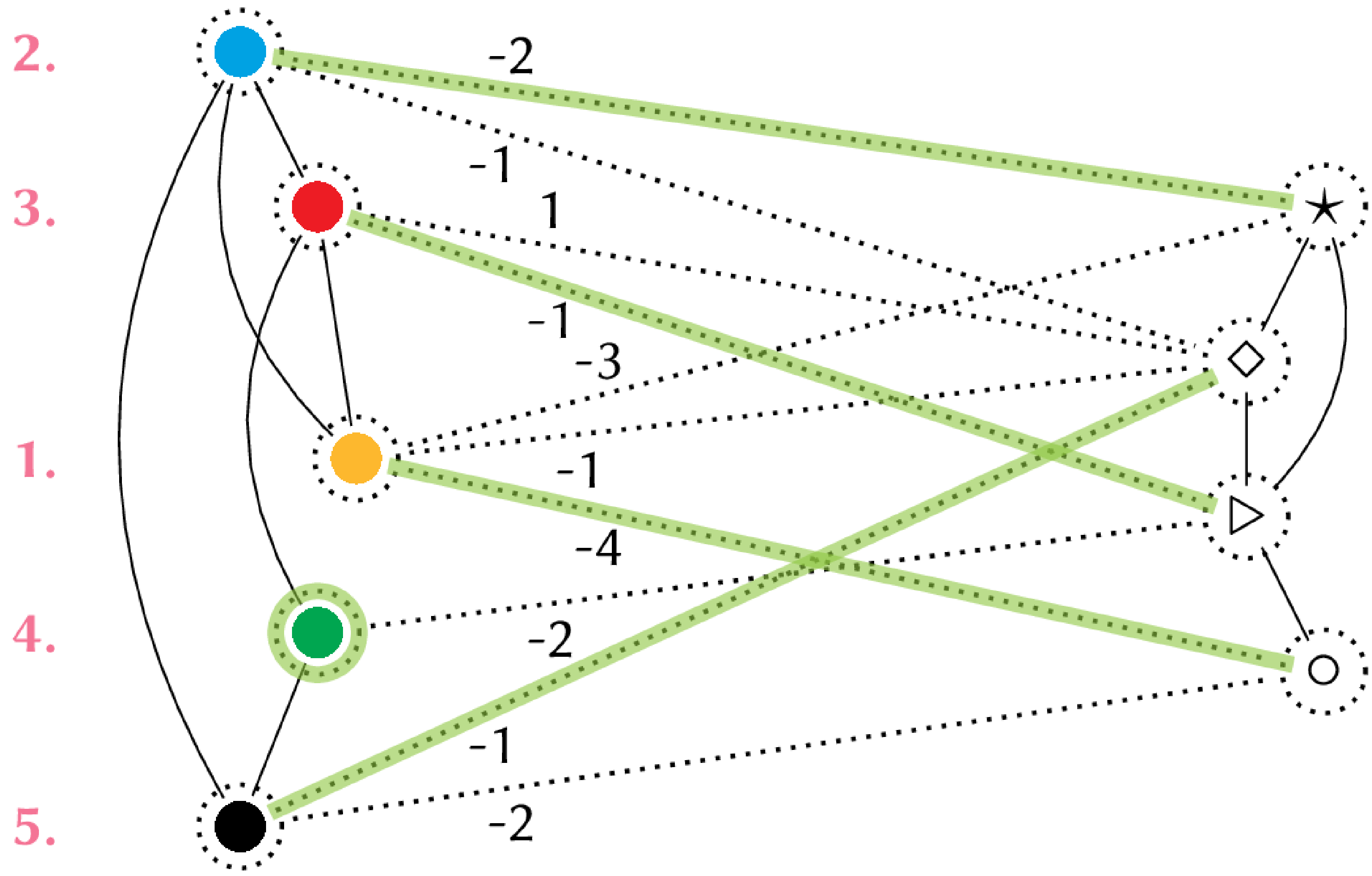
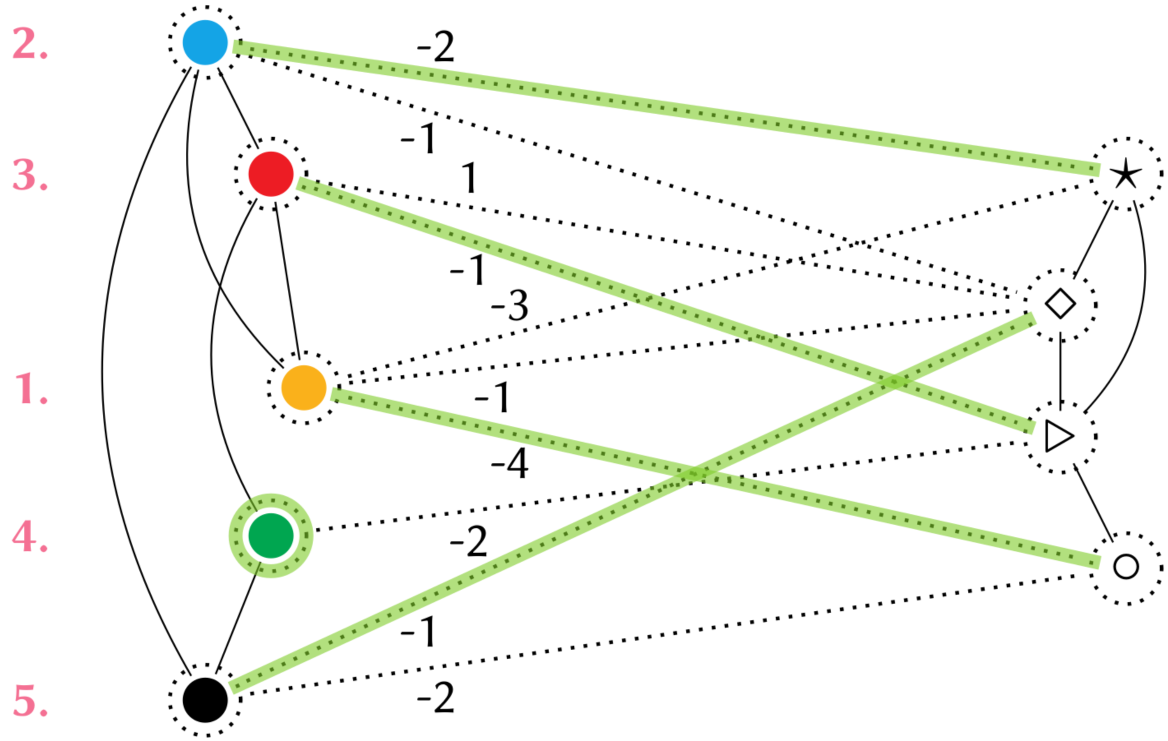
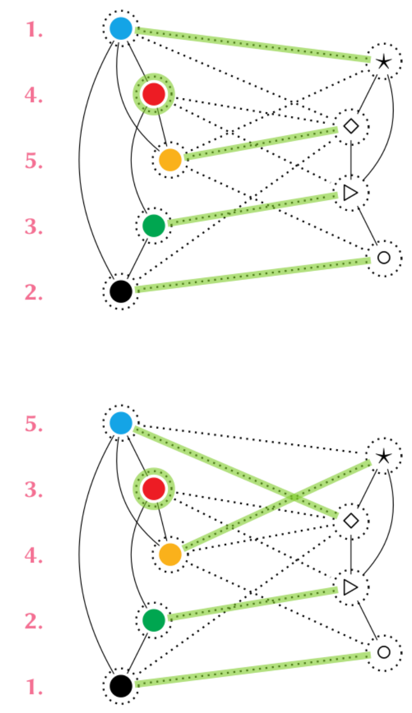


Randomized greedy heuristic





3. Strong primal heuristic: Fusion moves

1. Generate proposal solutions
2. Fuse it with the current one
3. Update the current solution
4. Goto 1.

- *Fusion moves* (fm, fm-bca)

L. Hutschenreiter et al. 2021, Fusion moves for graph matching

Design of the best performing methods

1. Based on integer linear program (**ILP**) **representation**

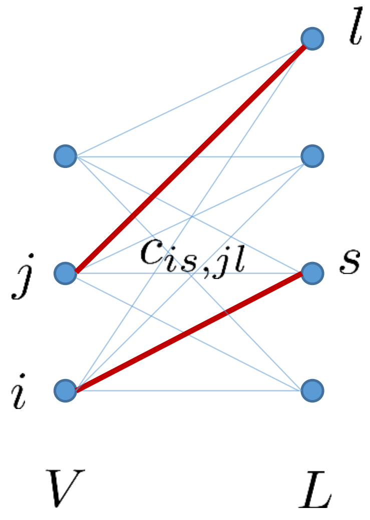
2. **Optimize Lagrange dual** and simplify the cost structure of the problem

3. Run a **strong primal heuristic** on the simplified costs

- *Fusion moves* (fm, fm-bca)

Integer linear program representation

1. ILP representation



$$\min_{x\in\{0,1\}^{V\times L}} \sum_{i,j\in V}\sum_{s,l\in L} c_{is,jl}x_{is}x_{jl}$$

$$\text{s.t.: } \sum_{s} x_{is} \leq 1 \ \forall i$$

$$\sum_{i} x_{is} \leq 1 \ \forall s$$

Integer **quadratic** program (IQP)

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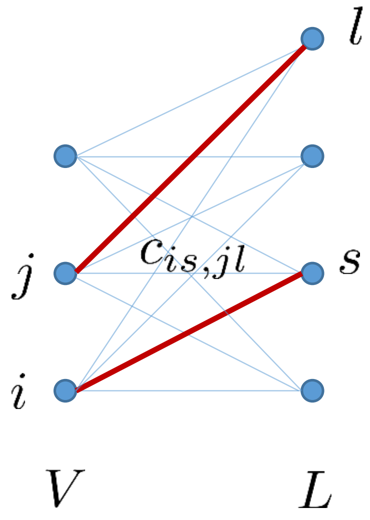

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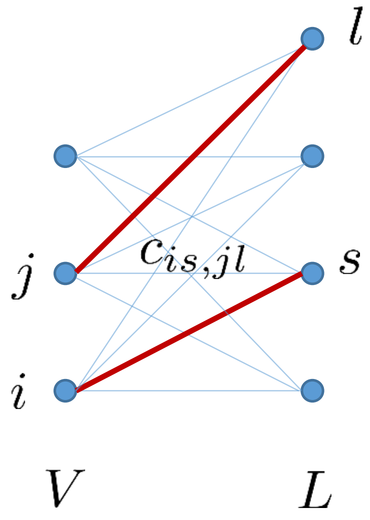
Integer **quadratic** program (IQP)

$$\min_{\substack{x \in \{0,1\}^N \\ Ax \leq b}} \langle c, x \rangle$$

Integer linear program (ILP)

- Well-studied, efficient off-the-shelf solvers (e.g. Gurobi)
- Powerful relaxations/approximative techniques exist

1. ILP representation: Variables lifting



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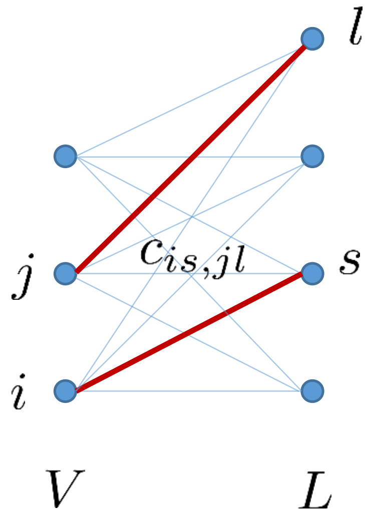

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Integer linear program (ILP)

Lifted variables / problem

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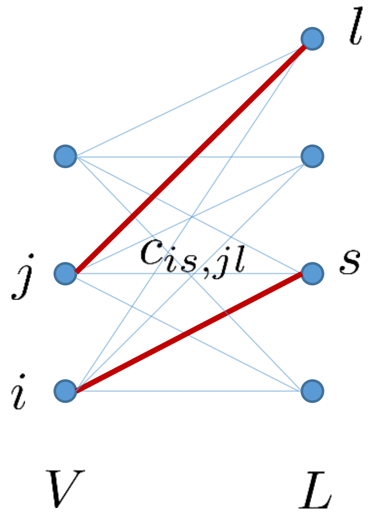

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Integer linear program (ILP)

Should hold only for binary variables:

x_{is}	1	1	0	0
x_{jl}	1	0	1	0
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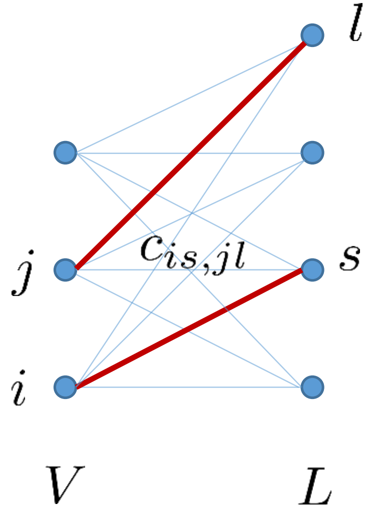

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Linear constraints sufficient for binary variables:

$$x_{iji'j'} \leq x_{ij}$$

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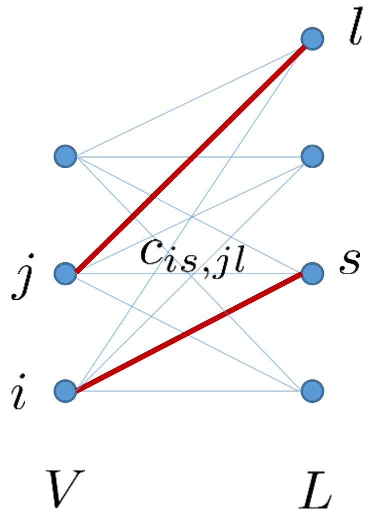
$$x_{iji'j'} \geq x_{ij} + x_{i'j'} - 1$$

1. ILP representation: Variables lifting



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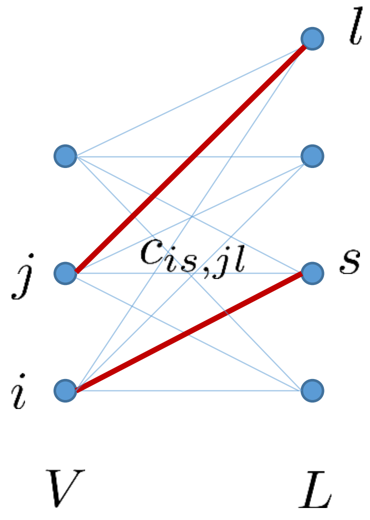

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Integer linear program (ILP)

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Different ILP representations/lifted constraints exist

2. Lagrange relaxation: Definition and meaning

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Reduced/reparametrized costs

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Reduced costs simplify the problem!

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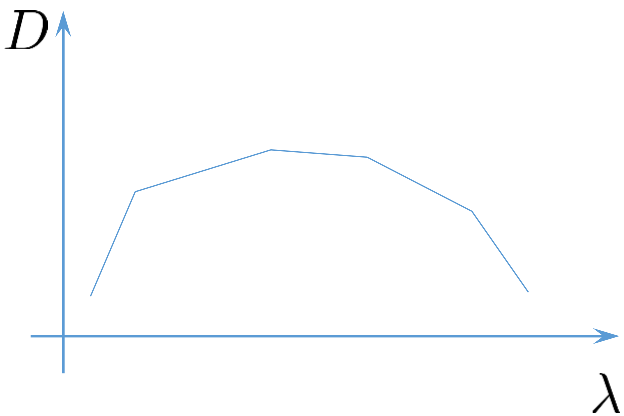

concave, piece-wise linear, large-scale

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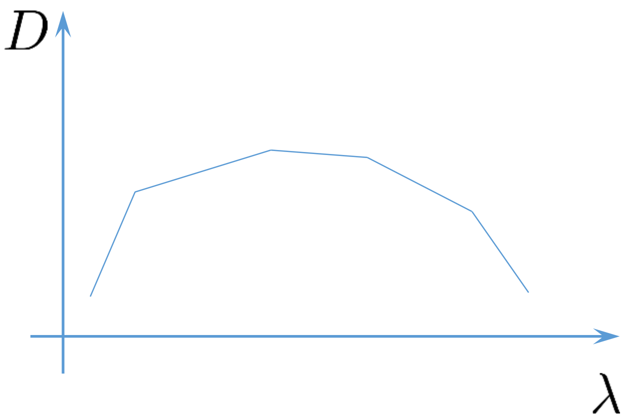

concave, piece-wise linear, large-scale

Large-scale => first-order methods:

- **Sub-gradient** [Shor 197X]
  - dd-lsX *Dual decomposition* [Torresani et al., 2013]
- Bundle [Kiwiel, Lemarechal 198X]
- Proximal [e.g. Parikh, Boyd 2013]
- **Block-coordinate ascent (BCA)** [196X]
  - fm-bca *Fusion Moves + BCA* [Hutschenreiter et al., 2021]
  - mp-XXX *Message passing* [Swoboda et al. 2017]
- Smoothing + BCA/accelerated gradient [Nesterov 200X]

Putting all together

1. Consider **ILP representation**

$$x_{is,jl} := x_{is} x_{jl}$$

2. Optimize Lagrange dual and
simplify the cost structure of the problem

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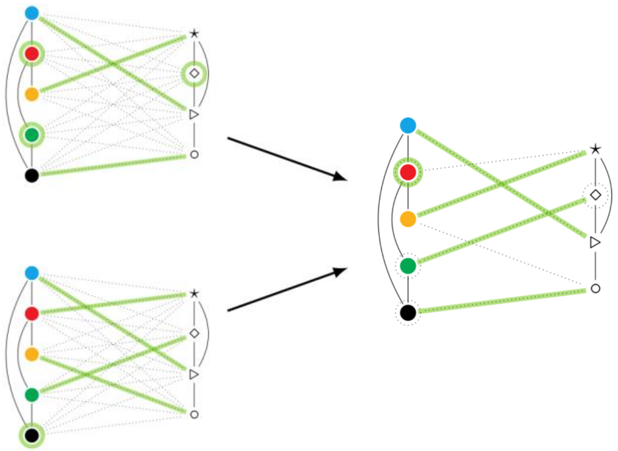

What expects us in future?