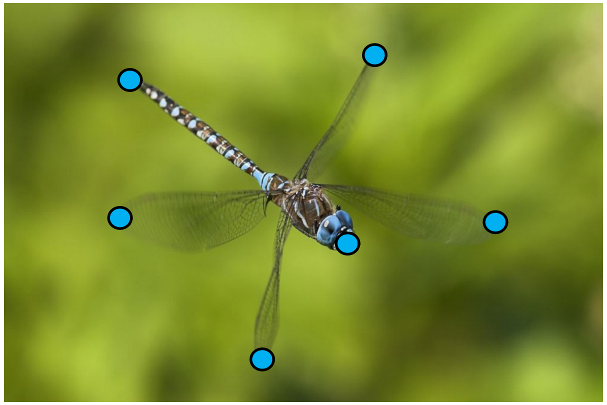
What is Graph- and Multi-Graph Matching
and
Why we need them

Bogdan Savchynskyy

27/09/2022

What is Graph-Matching?



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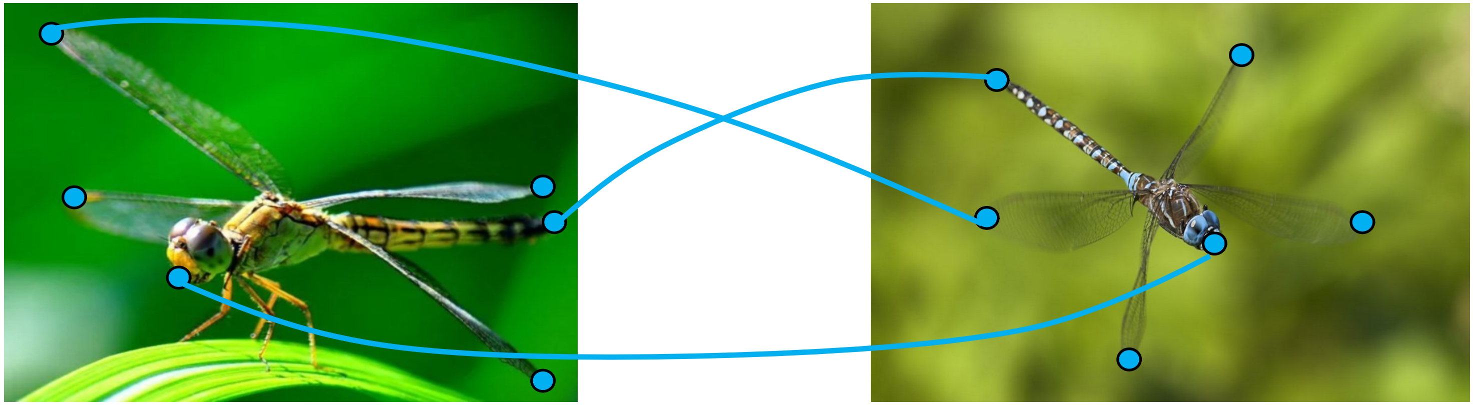
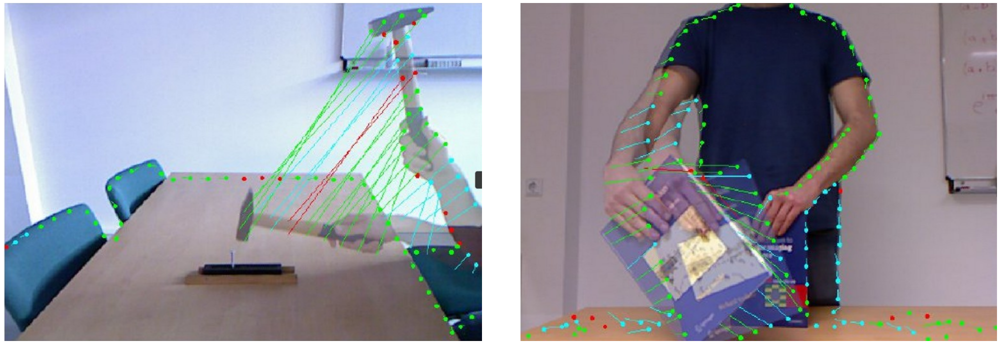


Graph matching, linear/quadratic assignment problem, weighted bipartite matching

Applications of (Multi-)Graph-Matching in CV

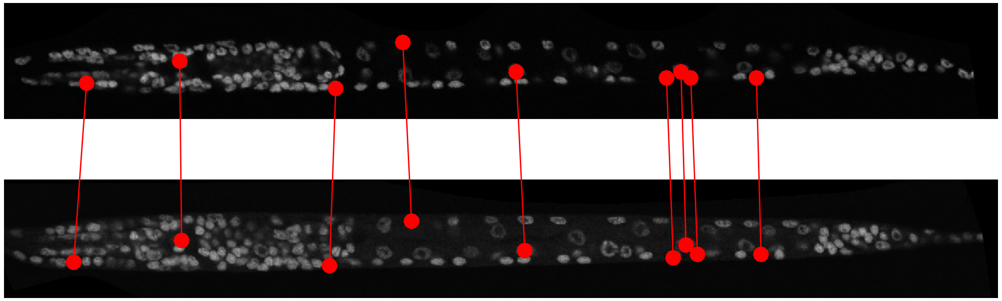
Keypoint matching of different objects

Image courtesy: [Rolinek et al. 2020]



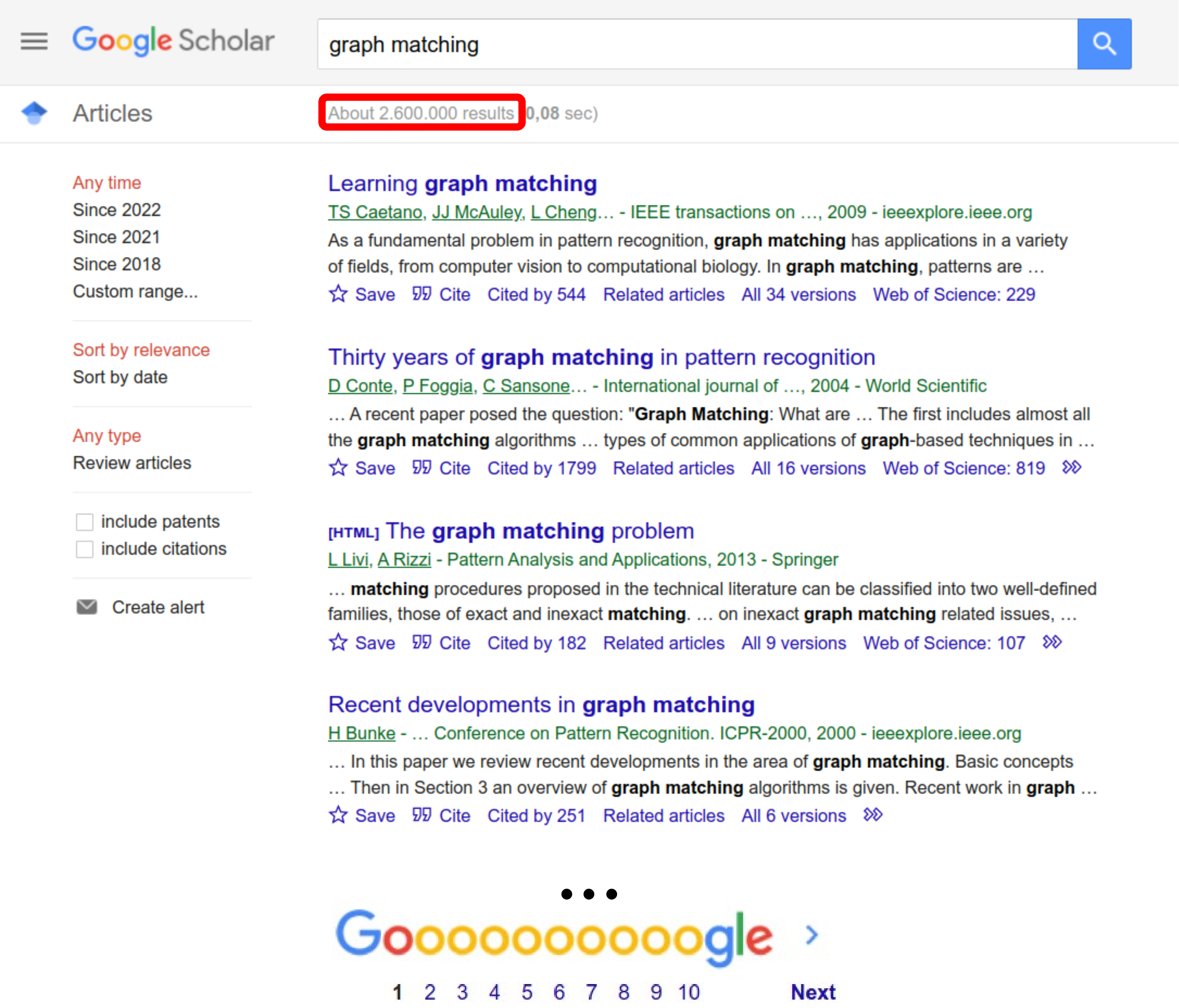
Non-rigid motion estimation

Image courtesy : [Alhaija et al. 2015]



Cell matching and tracking

Related work



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[HTML] The **graph matching** problem
L Livi, A Rizzi - Pattern Analysis and Applications, 2013 - Springer
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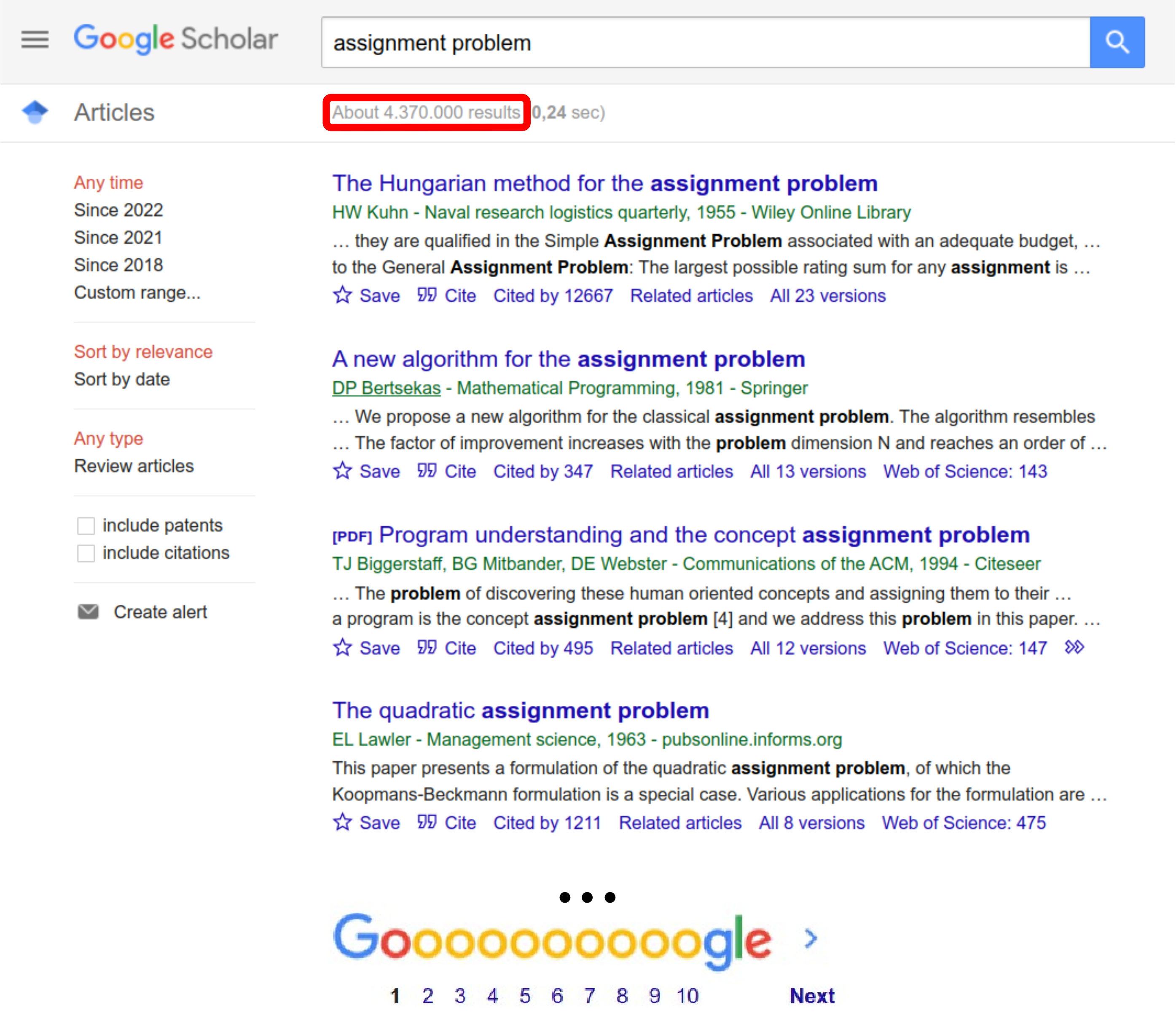
Related work



assignment problem

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The Hungarian method for the **assignment problem**
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TJ Biggerstaff, BG Mitbander, DE Webster - Communications of the ACM, 1994 - Citeseer
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• • •

Milestones and surveys: Very subjective!

Operations research:

1955: Kuhn. The Hungarian method for the assignment problem
1957: Koopmans and Beckmann. Assignment problems and the location of economic activities
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Computer Vision:

Milestones and surveys: Very subjective!

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2022: Haller et al. A comparative study of graph matching algorithms in computer vision

What we are going to address in the tutorial

13:30-14:00 What is graph- and multi-graph matching and why we need it

14:00-14:30 **Applications** in computer vision and bio-imaging

14:30-15:00 **Optimization** for graph matching and a recent comparison study

15:00 – 15:30 – *Coffee break*

15:30-16:15 Deep graph matching: **Learning** to match

~~16:15-17:00 Multi-graph matching: **Optimization** methods and **applications**~~

Outline (of this particular talk)

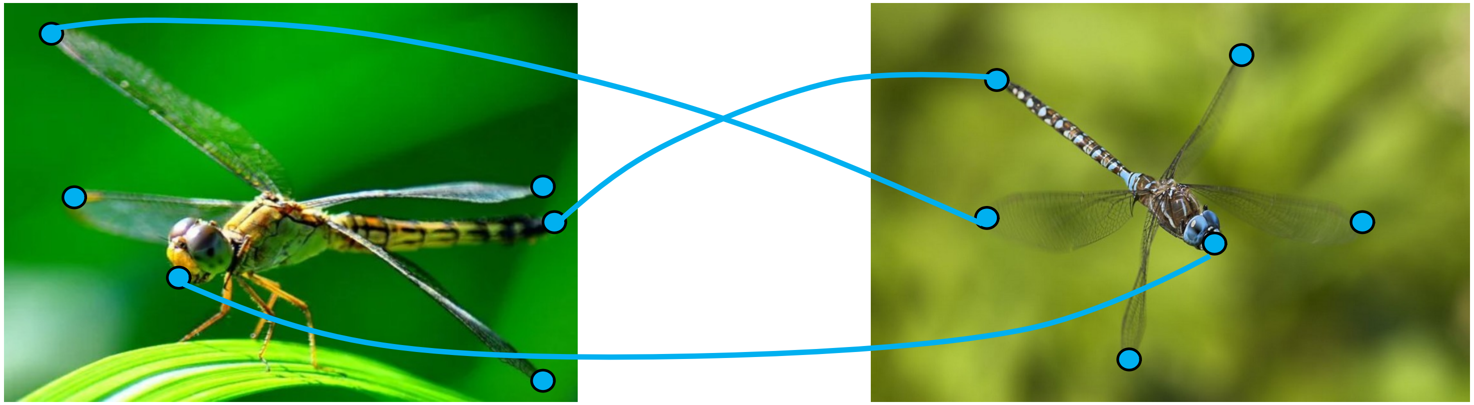
Linear and quadratic assignment problems
- Expressing power
- Integer programs
- Complexity

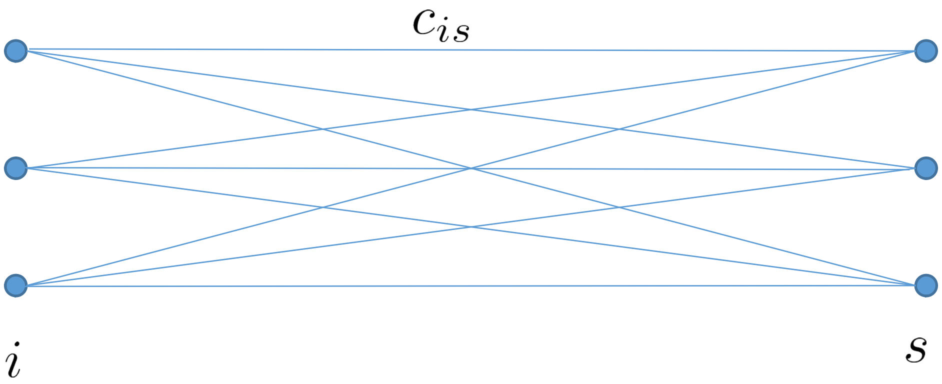
Where the name **graph matching** comes from?
- Classical formulation
- Koopmans-Beckmann form

How different is it in Computer Vision vs. Operations Research?
- Outliers and (in)complete matching
- Speed vs. precision

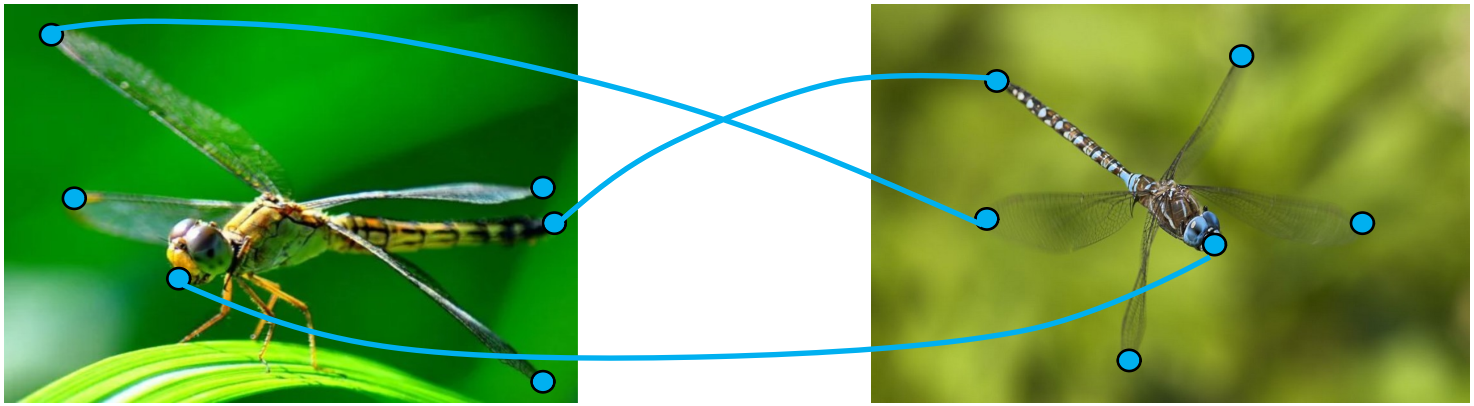
Multigraph matching
- Definition and complexity

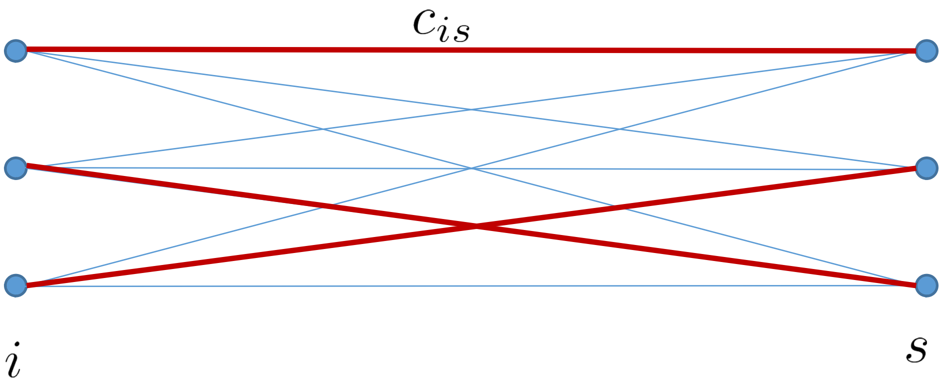
Linear assignment problem





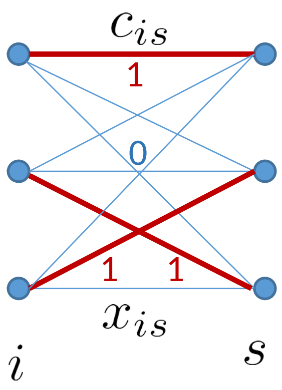
Linear assignment problem





Another name: Weighted bipartite matching

Linear assignment problem

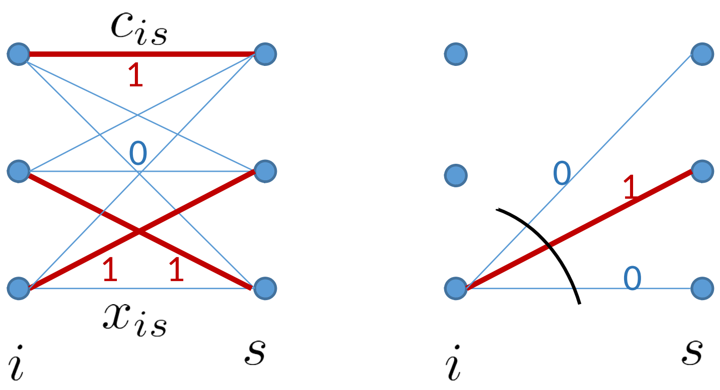


c_{is} - cost of matching $i \leftrightarrow s$

$x_{is} \in \{0, 1\}$ - edge selectors

$\sum_{is} c_{is}x_{is}$ - total matching cost

Linear assignment problem



$$i : \sum_{s=1}^{n} x_{is} = 1$$

c_{is} - cost of matching $i \leftrightarrow s$

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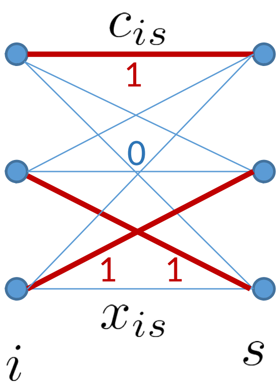

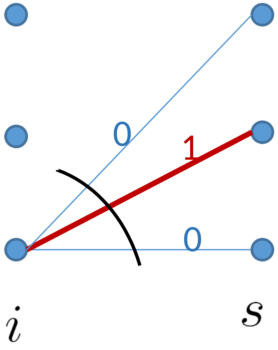

$$i: \sum_{s=1}^{n} x_{is} = 1$$

$$\min_{x \in \{0,1\}^{n \times n}} \sum_{i=1}^{n} \sum_{s=1}^{n} c_{is} x_{is}$$

$$\text{s.t.:} \sum_{s=1}^{n} x_{is} = 1 \ \forall i$$

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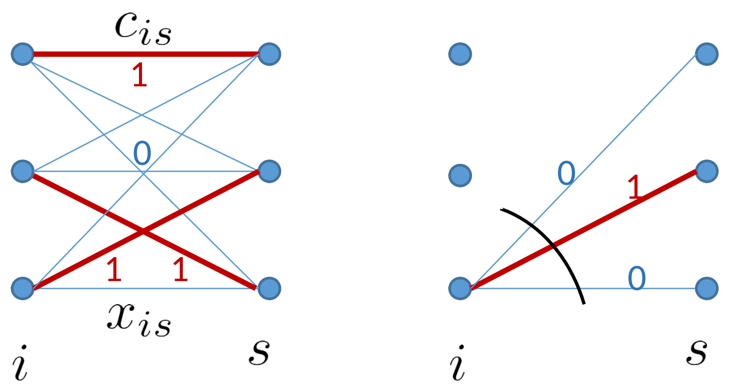
Linear assignment problem



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$$\text{s.t.:} \sum_{s=1}^{n} x_{is} = 1 \ \forall i$$

$$\sum_{i=1}^{n} x_{is} = 1 \ \forall s$$

$$= \min_{x \geq 0} \sum_{i=1}^{n} \sum_{s=1}^{n} c_{is} x_{is}$$

$$\text{s.t.:} \sum_{s=1}^{n} x_{is} = 1 \ \forall i$$

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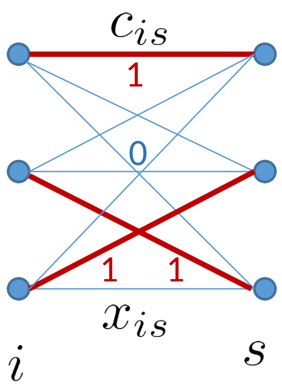
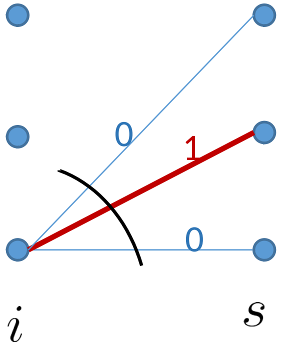
Linear assignment problem





$$i: \sum_{s=1}^{n} x_{is} = 1$$

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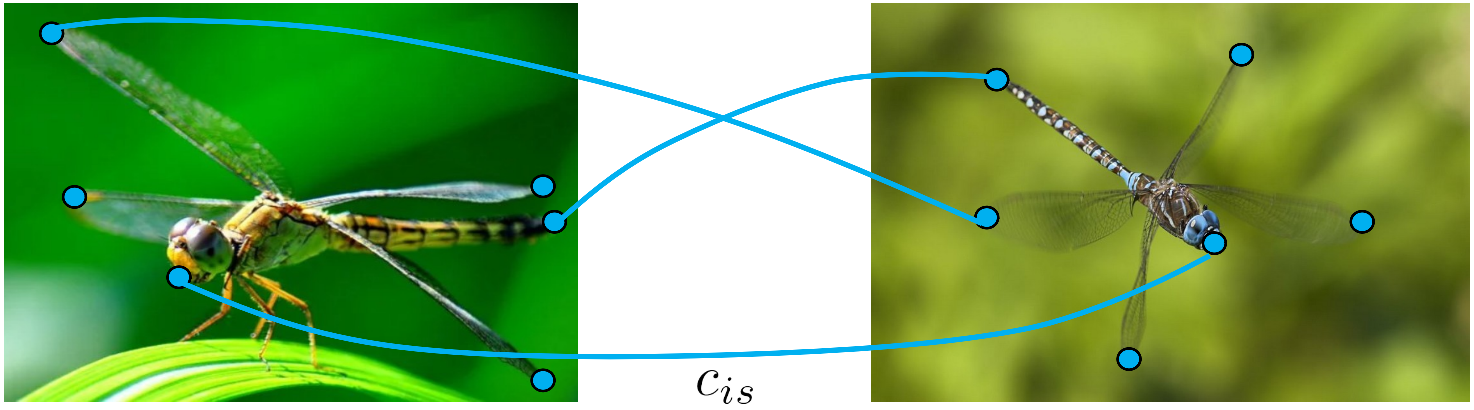
$\sum_{is} c_{is}x_{is}$ - total matching cost

$$= \boxed{\begin{aligned} &\min_{x\geq 0} \sum_{i=1}^{n}\sum_{s=1}^{n} c_{is}x_{is} \\ &\text{s.t.:} \sum_{s=1}^{n} x_{is} = 1\ \forall i \\ &\sum_{i=1}^{n} x_{is} = 1\ \forall s \end{aligned}}$$

Efficiently solvable e.g. by Hungarian method $O(n^3)$

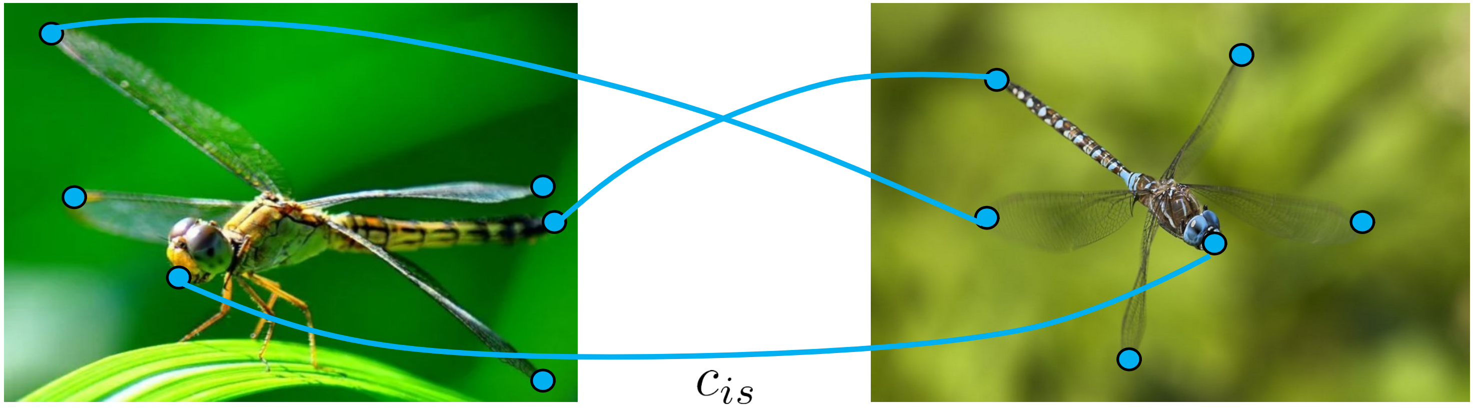
Python: *scipy.optimize.linear_sum_assignment()*

Quadratic assignment problem



appearance cost only, no geometry!

Quadratic assignment problem



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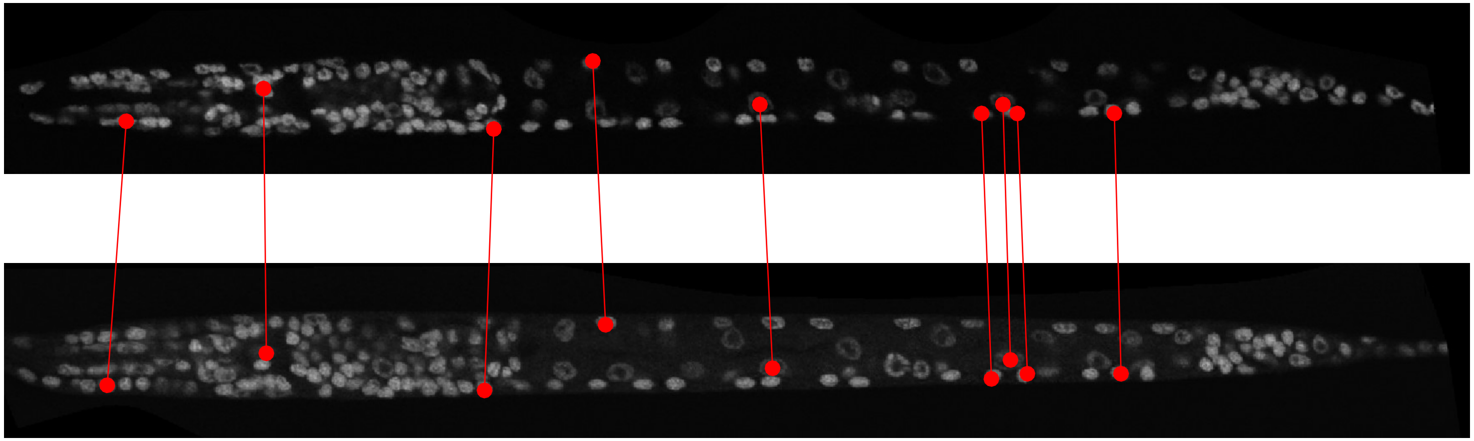
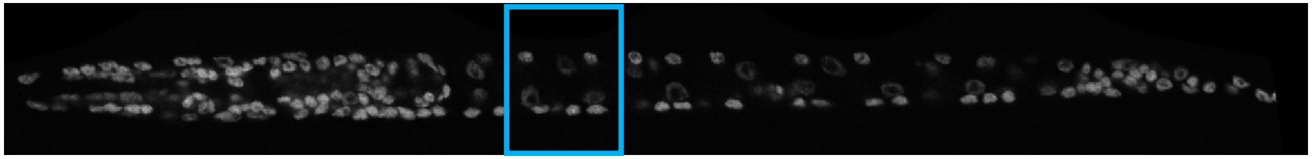
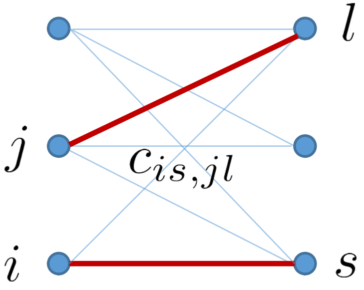
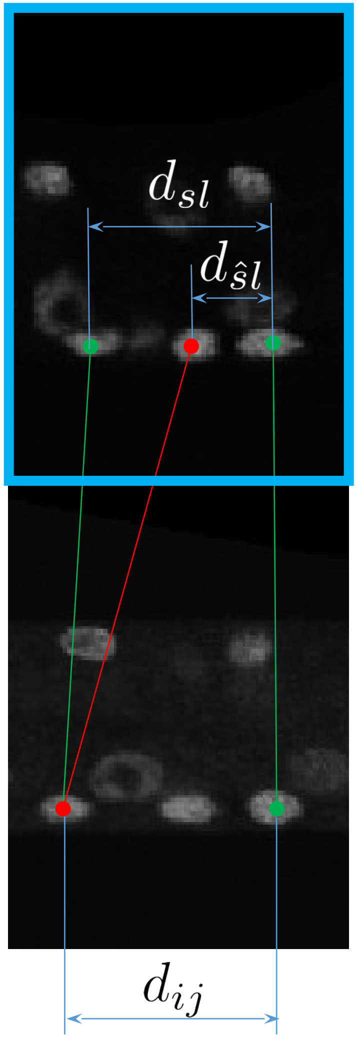


Very similar appearance, geometry is crucial!

Quadratic assignment problem

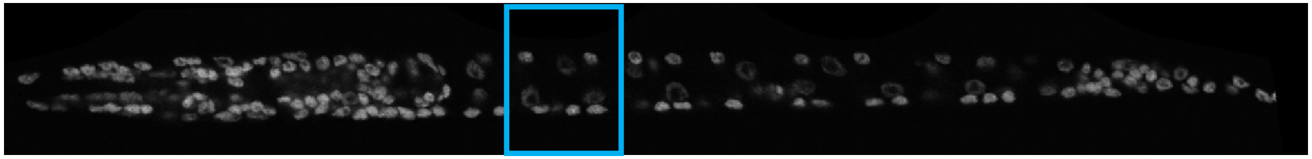






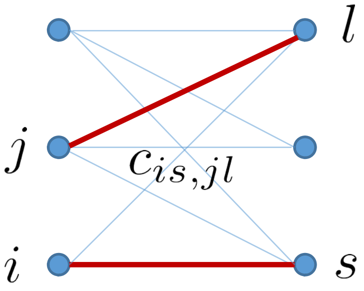
$$c_{is,jl} := (d_{ij} - d_{sl})^2$$

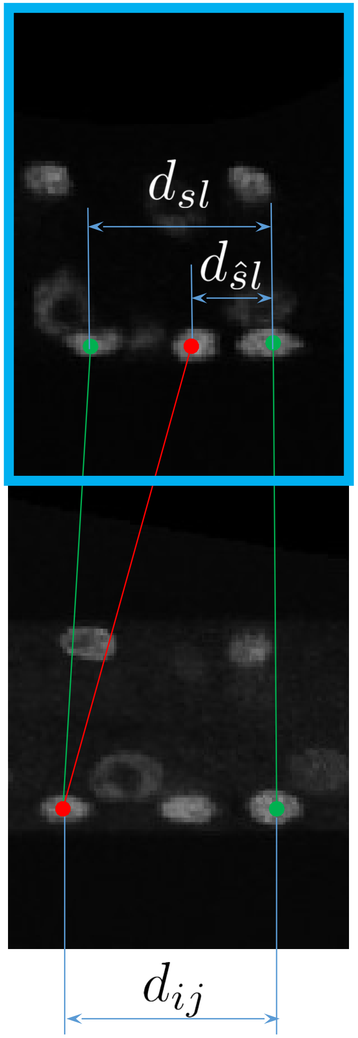
Quadratic assignment problem



Total matching cost:

$$\sum_{is} c_{is}x_{is} + \sum_{\substack{i\neq j\\ s\neq l}} c_{is,jl}x_{is}x_{jl}$$





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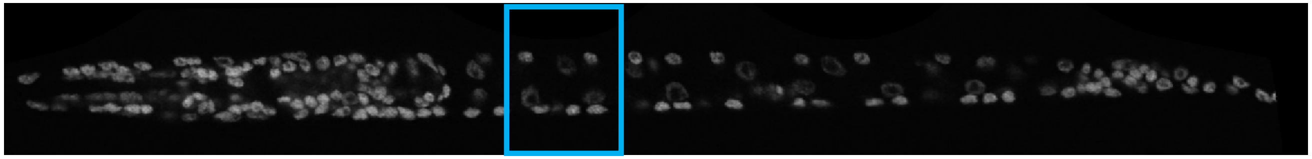

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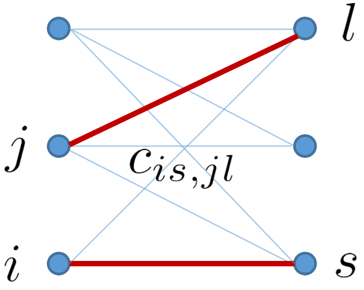

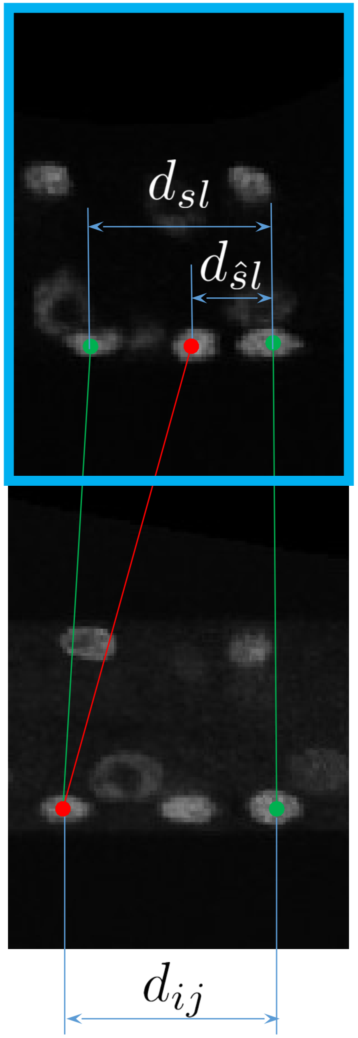

$c_{is} := c_{is,is}$ - unary costs

$$\min_{x \in \{0,1\}^{n \times n}} \sum_{ijsl} c_{is,jl} x_{is} x_{jl}$$

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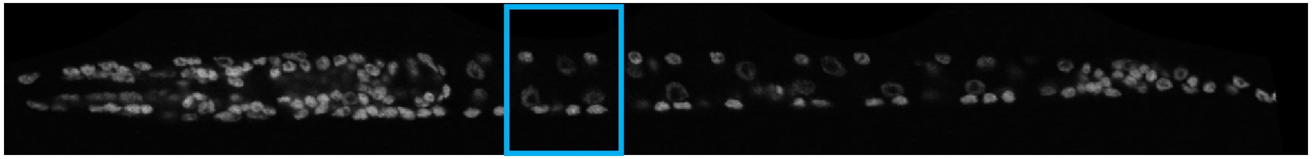
Quadratic assignment problem



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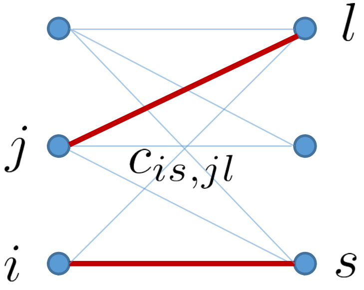

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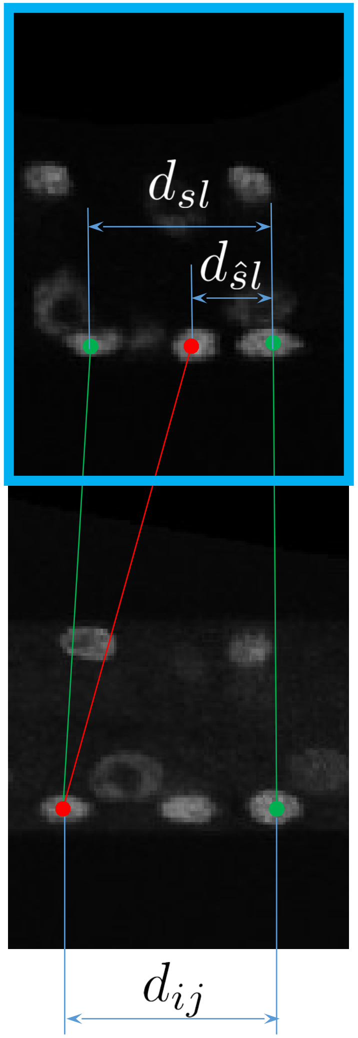
NP-hard



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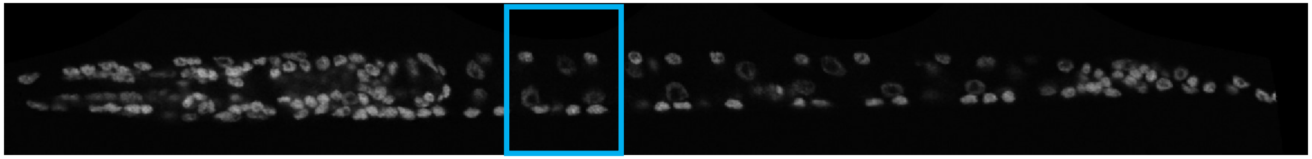
Quadratic assignment problem



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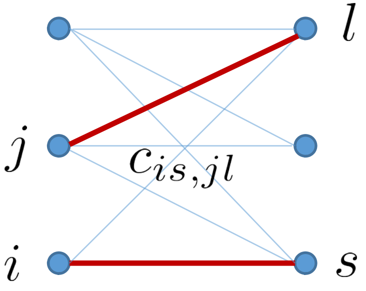

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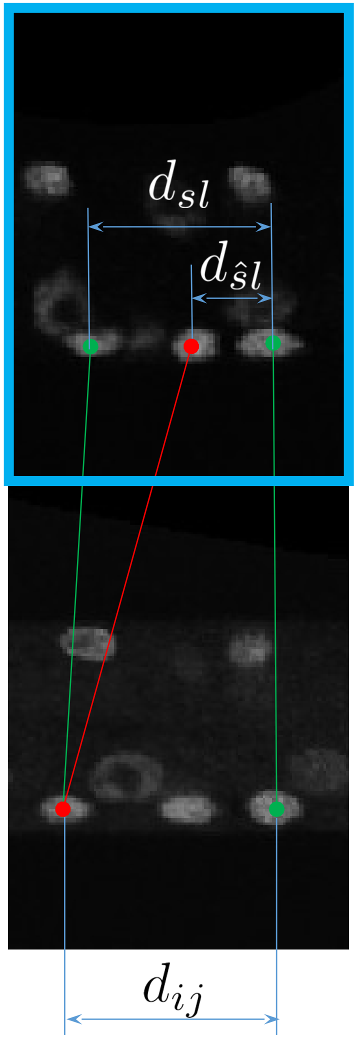
$$\text{s.t.:} \sum_{s} x_{is} = 1 \ \forall i$$

$$\sum_{i} x_{is} = 1 \ \forall s$$

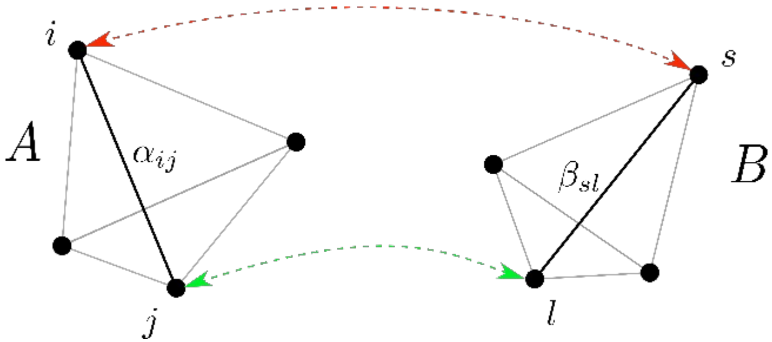
$$\Longrightarrow \quad \min_{x \in P} x^{\top} C x$$

NP-hard



$$c_{is,jl} := (d_{ij} - d_{sl})^2$$

Where the name **Graph Matching** comes from?



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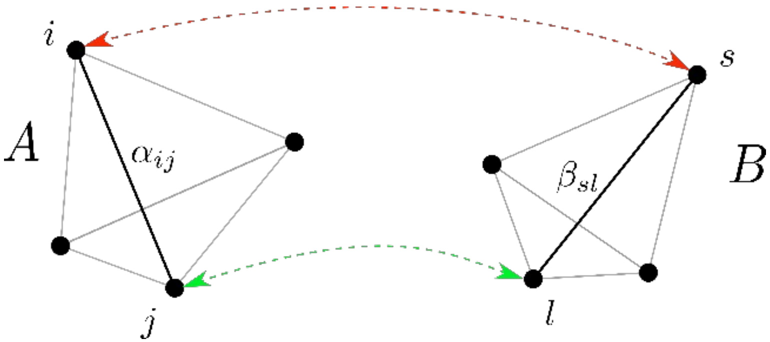

$$c_{is,jl} := f(\alpha_{ij}, \beta_{sl})$$

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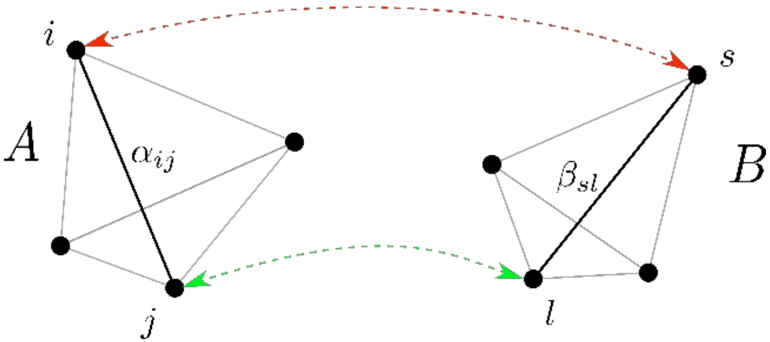

$$c_{is,jl} := f(\alpha_{ij}, \beta_{sl})$$

$$c_{is,jl} := (\alpha_{ij} - \beta_{sl})^2$$

“classical” graph matching

$$c_{is,jl} = \alpha_{ij}\beta_{sl}$$

Koopmans-Beckmann form

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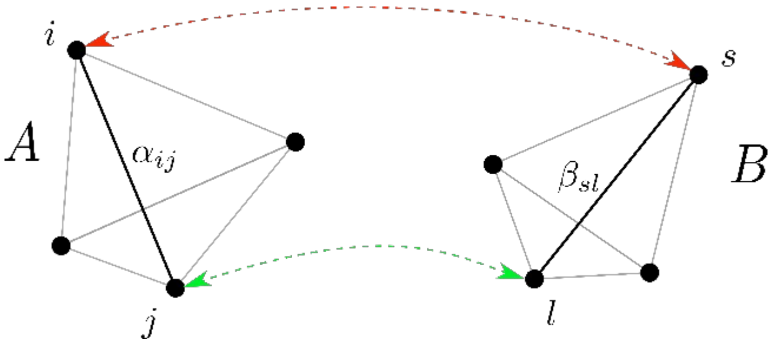

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$$c_{is,jl} := (\alpha_{ij} - \beta_{sl})^2$$

“classical” graph matching

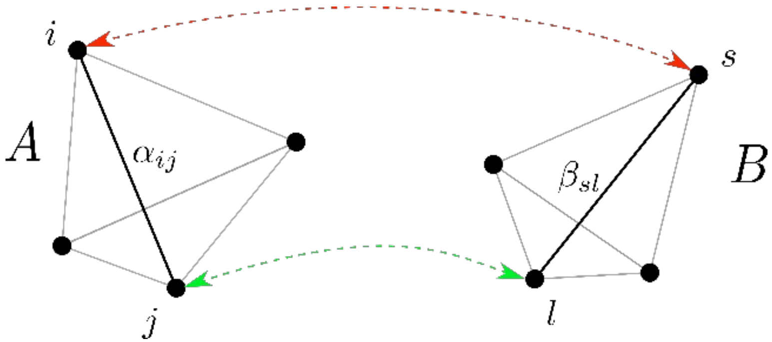
$$c_{is,jl} = \alpha_{ij}\beta_{sl}$$

Koopmans-Beckmann form

$$c_{is,jl} = \cancel{\alpha_{ij}\beta_{sl}}$$

Lawler form

Where the name **Graph Matching** comes from?



$$c_{is,jl} := f(\alpha_{ij}, \beta_{sl})$$

$$c_{is,jl} := (\alpha_{ij} - \beta_{sl})^2$$

"classical" graph matching

$$c_{is,jl} = \alpha_{ij}\beta_{sl}$$

Koopmans-Beckmann form

$$c_{is,jl} = \cancel{\alpha_{ij}\beta_{sl}}$$

Lawler form

[Zhou, De La Torre 2012, Factorized Graph Matching]

Factorization into a sum of Koopmans-Beckmann forms

Wait…

Do we still need optimization in era of NNs?

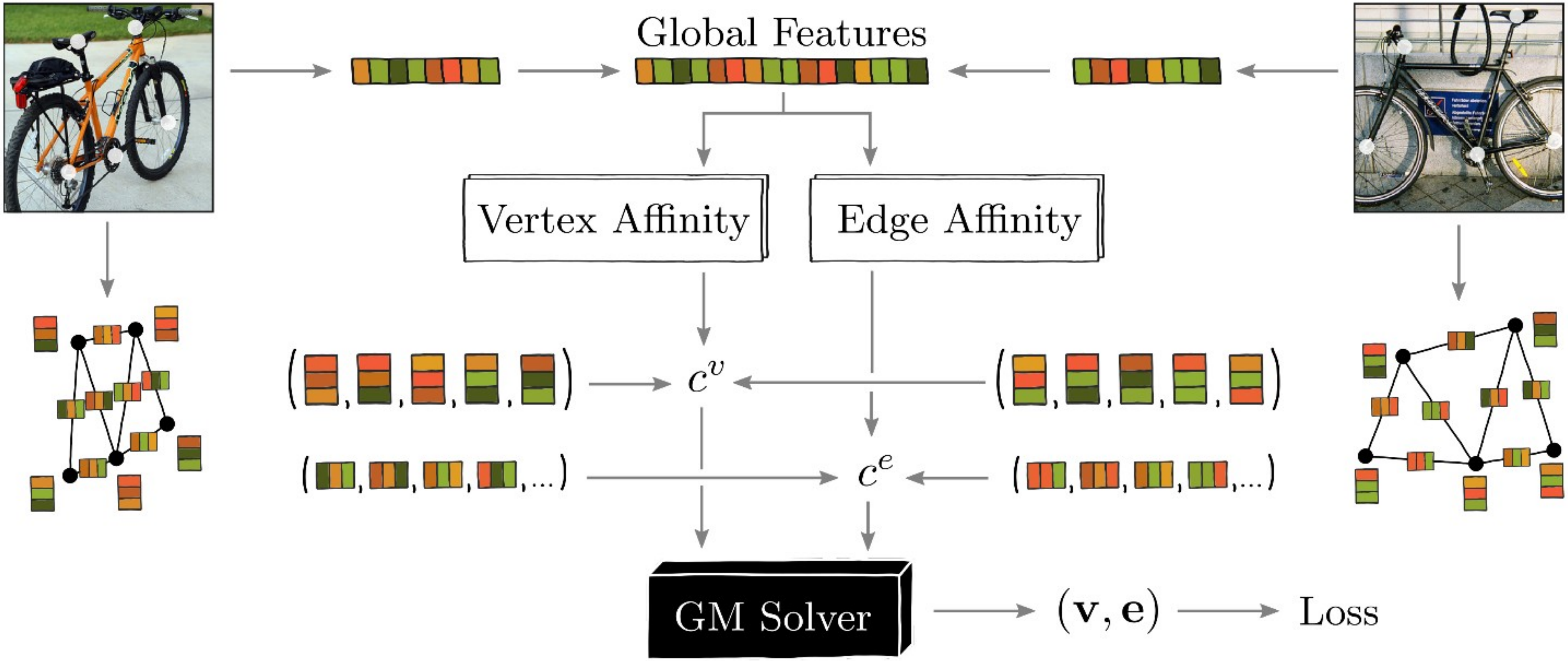
(except for learning NNs)

Wait…

Yes, we do:

[**Vlastelica et al. ’19 Differentiation of blackbox combinatorial solvers**]

State-of the art deep graph matching method

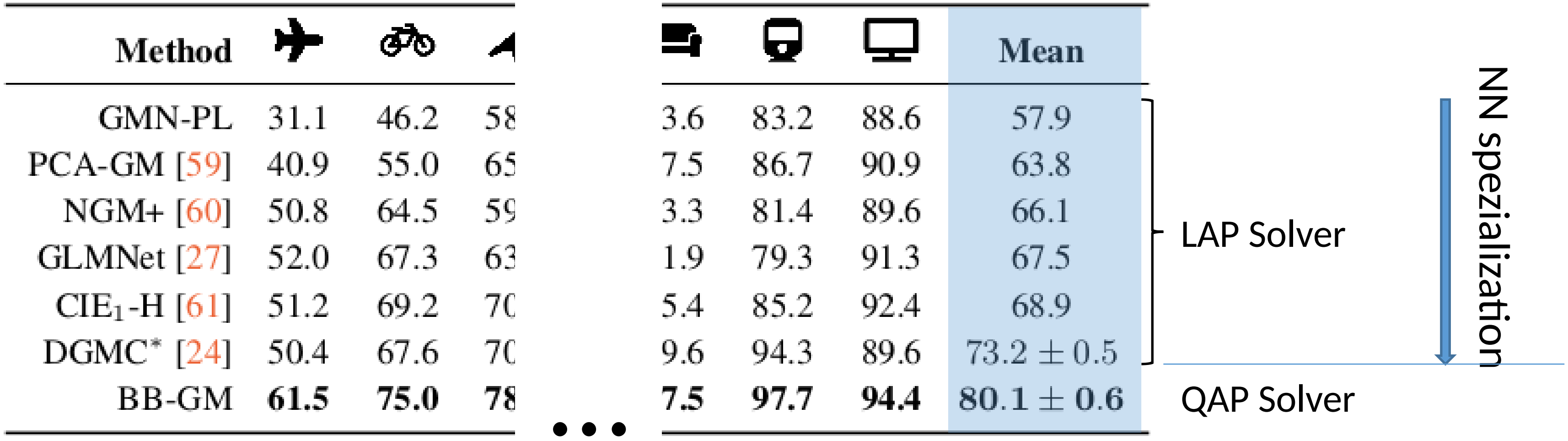


Network Architecture

[Rolinek et al. ’20 Deep graph matching via black-box differentiation of combinatorial solvers]

Do we still need optimization in era of NNs?

PASCAL VOC Feature points matching:



Method	✈	🚲				🚆	🖥	Mean
GMN-PL	31.1	46.2	58		3.6	83.2	88.6	57.9
PCA-GM [59]	40.9	55.0	65		7.5	86.7	90.9	63.8
NGM+ [60]	50.8	64.5	59		3.3	81.4	89.6	66.1
GLMNet [27]	52.0	67.3	63		1.9	79.3	91.3	67.5
CIE$_1$-H [61]	51.2	69.2	70		5.4	85.2	92.4	68.9
DGMC* [24]	50.4	67.6	70		9.6	94.3	89.6	73.2 ± 0.5
BB-GM	**61.5**	**75.0**	**78**	• • •	**7.5**	**97.7**	**94.4**	$\mathbf{80.1 \pm 0.6}$

LAP Solver

QAP Solver

NN spezialization

Everyone uses at least a LAP solver!

[Rolinek et al. ’20 Deep graph matching via black-box differentiation of combinatorial solvers]

Bi-Stochastic layer: "Differentiable" linear assignment

$$\min_{x \geq 0} \quad \sum_{i=1}^{n} \sum_{s=1}^{n} c_{is} x_{is} - \rho \mathrm{H}(\mathrm{x})$$

$$\text{s.t.:} \sum_{s=1}^{n} x_{is} = 1 \ \forall i$$

$$\sum_{i=1}^{n} x_{is} = 1 \ \forall s$$

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$$\sum_{i=1}^{n} x_{is} = 1 \ \forall s$$

smooth, differentiable

differentiable Sinkhorn algorithm

Sinkhorn algorithm:

Init $x_{is}^{0} := \exp(-c_{is}/T)$

Iterate:

1. Normalize each row
2. Normalize each column

$$\begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \cdots & x_{n,n} \end{pmatrix} \Big/ \textstyle\sum_{s=1}^{n} x_{2,s}$$

$$\Big/ \textstyle\sum_{i=1}^{n} x_{i,2}$$

Operations Research vs. Computer Vision

Operations Research:

- Koopmans-Beckman form (QAPLIB)
  $c_{is,jl} = \alpha_{ij}\beta_{sl}$

Computer Vision:

- Lawler form

Operations Research vs. Computer Vision

Operations Research:

- Koopmans-Beckman form (QAPLIB)
  $c_{is,jl} = \alpha_{ij}\beta_{sl}$
- C rather dense

Computer Vision:

- Lawler form
- C often sparse
  $c_{is} = \infty \text{ and } c_{is,jl} = 0$

Operations Research vs. Computer Vision

Operations Research:

- Koopmans-Beckman form (QAPLIB)
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- Speed

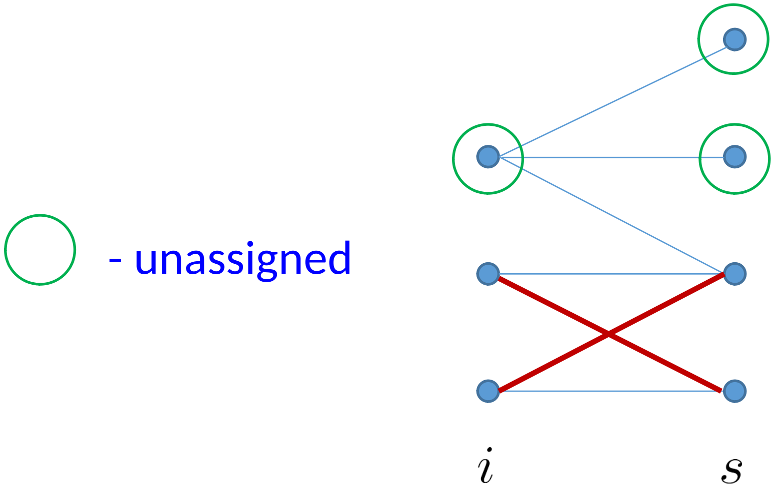
Operations Research vs. Computer Vision

Operations Research:

- Koopmans-Beckman form (QAPLIB)
  $c_{is,jl} = \alpha_{ij}\beta_{sl}$
- C rather dense
- Precision
- Complete assignment

Computer Vision:

- Lawler form
- C often sparse
  $c_{is} = \infty$ and $c_{is,jl} = 0$
- Speed
- Incomplete assignment

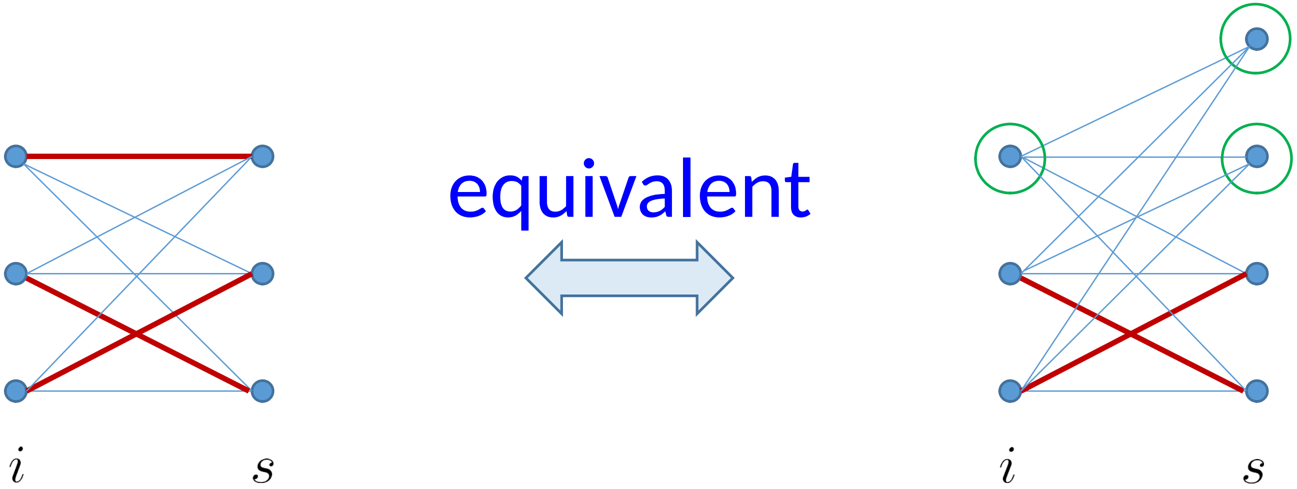


$$\min_{x\in\{0,1\}^{n\times n}} \sum_{ijsl} c_{is,jl} x_{is} x_{jl}$$

$$\text{s.t.:} \sum_{s} x_{is} \leq 1 \ \forall i$$

$$\sum_{i} x_{is} \leq 1 \ \forall s$$

Incomplete vs. complete assignment



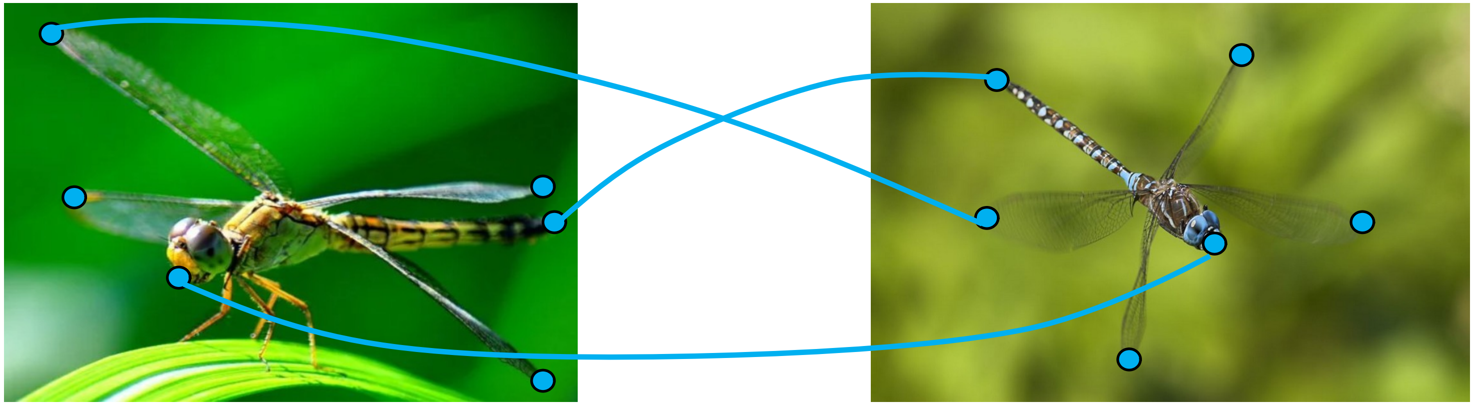
equivalent

complete
assignment

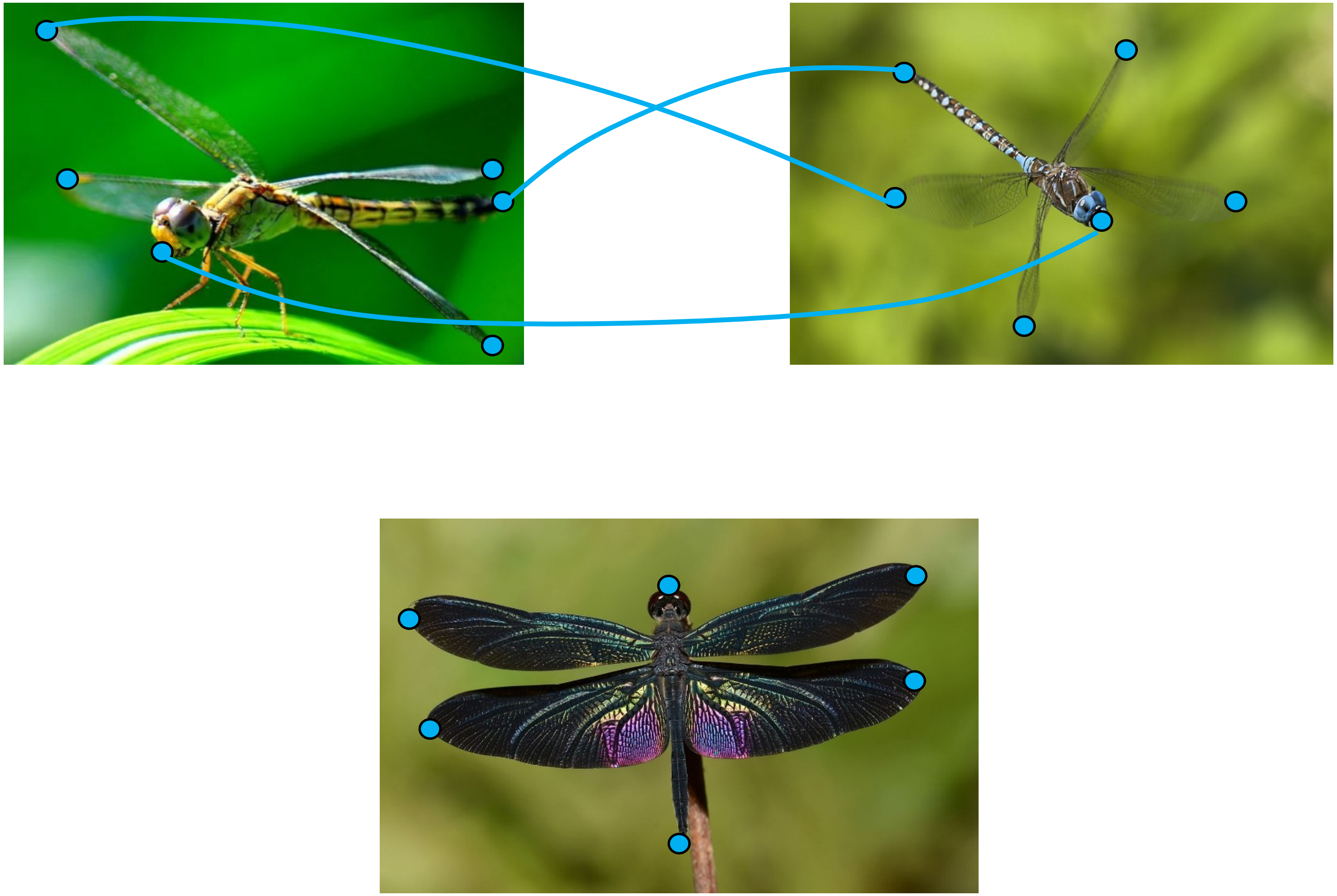
incomplete
assignment

e.g. [Haller et al. 2022. A comparative study of graph matching algorithms in computer vision]

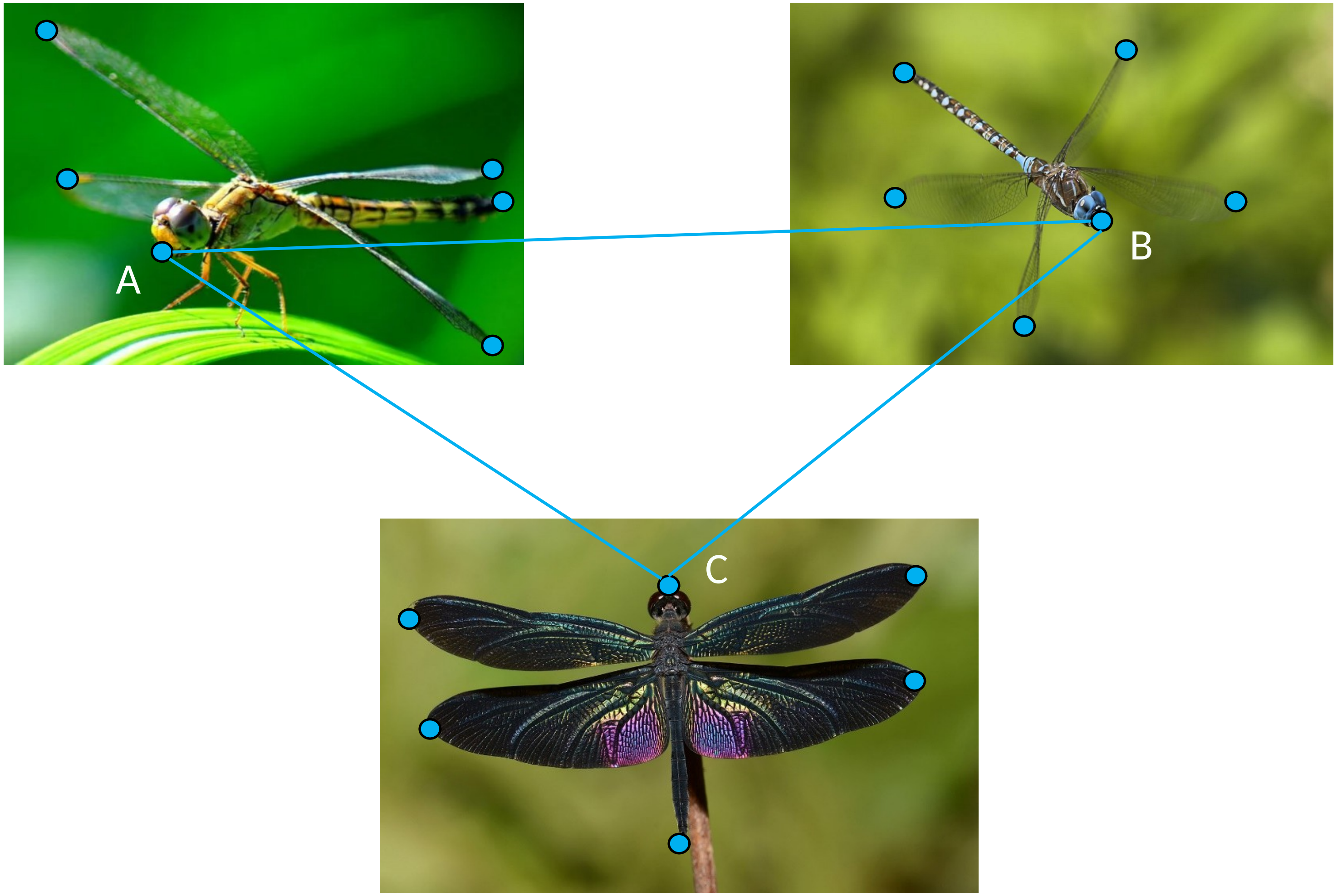
What is Multi-Graph-Matching?



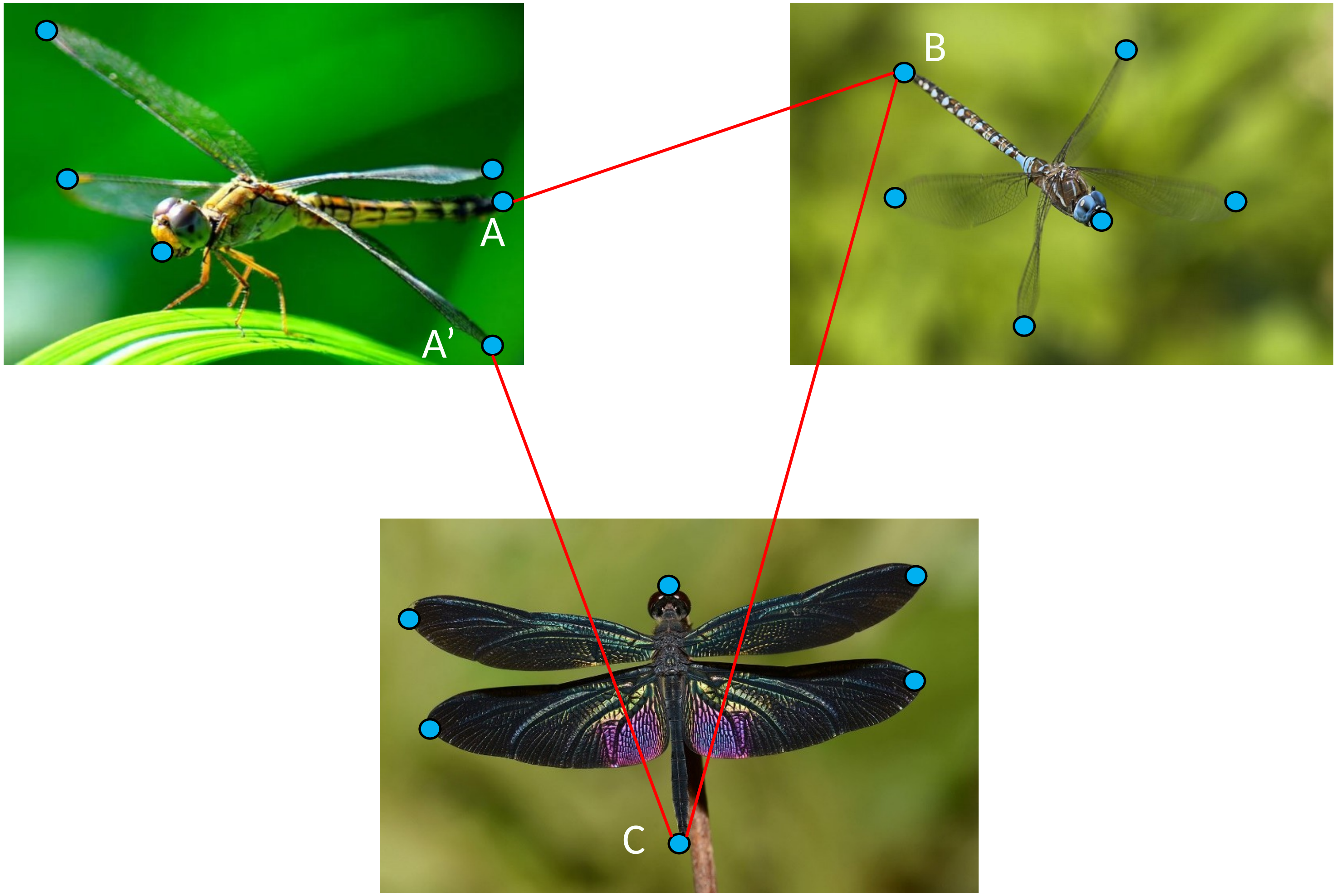
What is Multi-Graph-Matching?



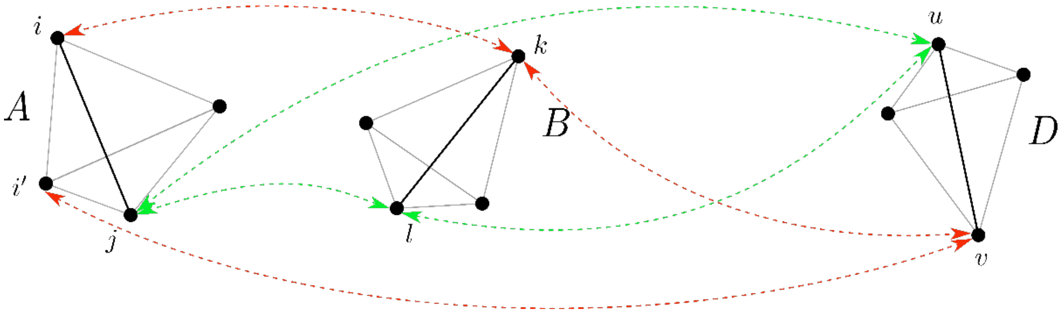
What is Multi-Graph-Matching?



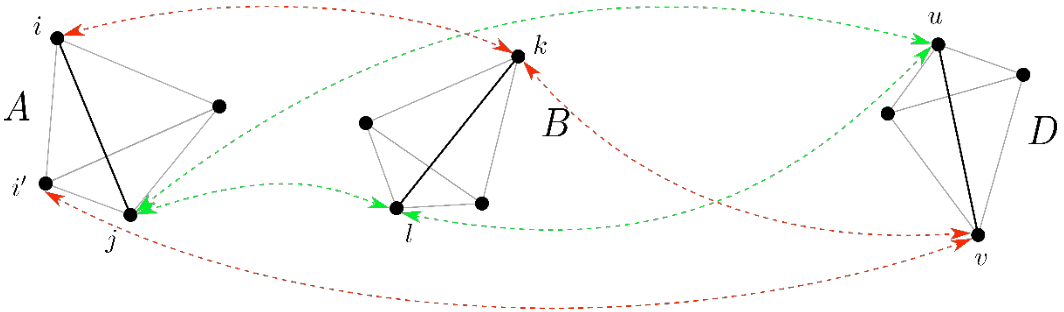
What is Multi-Graph-Matching?



Multigraph matching



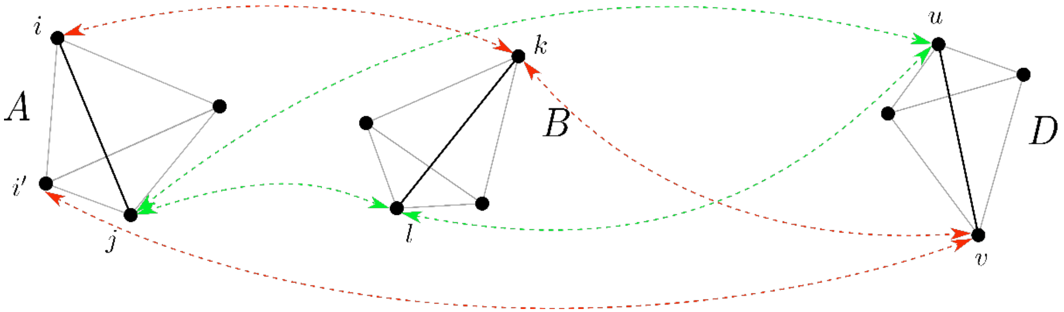
Multigraph matching



$[d] = [1, \dots, d]$ - set of graphs
$C^{[pq]}$ - cost matrix for $p \leftrightarrow q$ graphs
$x^{[pq]}, X^{[pq]}$ - $p \leftrightarrow q$ assignment

Multigraph matching

$$\boxed{\text{Recall QAP: } \min_{x \in P} x^\top C x}$$



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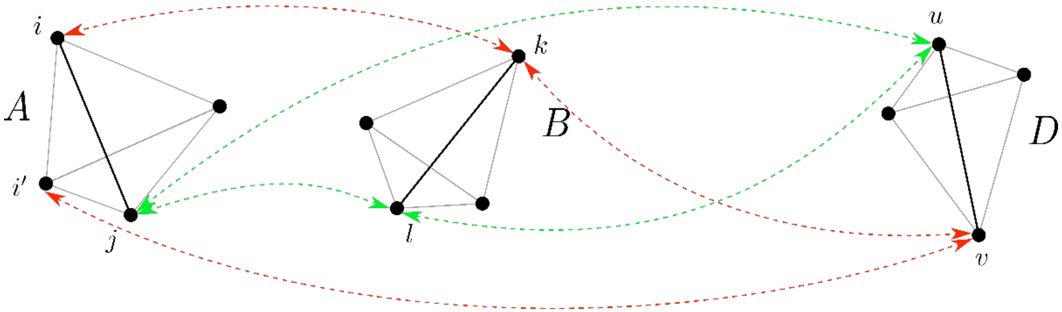

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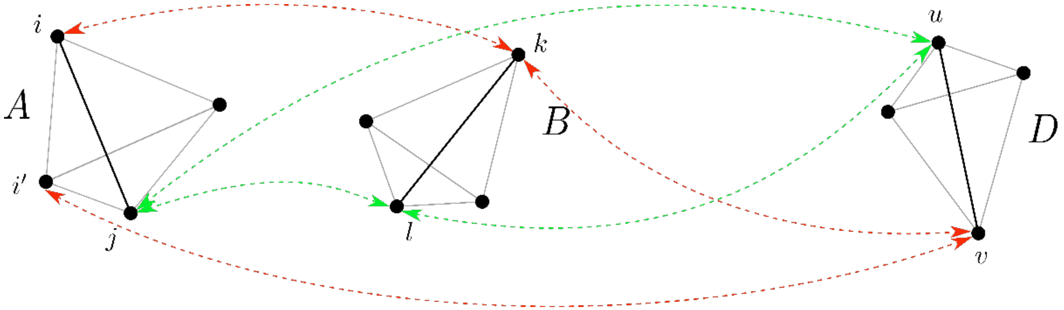
$$\min_{X \in P} \sum_{p,q \in [d]} (x^{[pq]})^\top C^{[pq]} x^{[pq]}$$

$$\text{s.t.} \quad X^{[pq]} X^{[qr]} \leq X^{[pr]} \quad \forall p, q, r \in [d]\,.$$

Multigraph matching

Recall QAP: $\min_{x \in P} x^\top C x$



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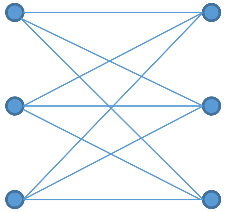
$$\text{s.t.} \quad X^{[pq]} X^{[qr]} \leq X^{[pr]} \quad \forall p, q, r \in [d] \,.$$

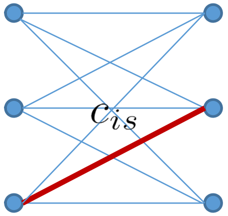
$\#$ of $[pq]$ pairs - $O(d^2)$
$\#$ of costs $c^{[pq]}_{is,jl}$ - $O(d^2 n^4)$
$\#$ of cycle consistency constraints - $O(d^3 n^3)$

Take home messages

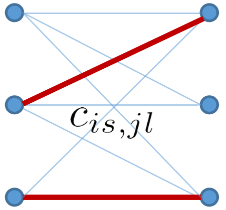
- Graph matching = matching finite point sets



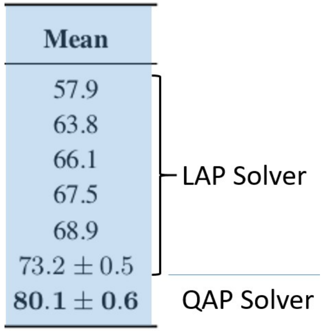
- Linear assignment – point features only



- Quadratic assignment – features related to a pair of points



- Optimization is a key component



Mean	
57.9	LAP Solver
63.8	
66.1	
67.5	
68.9	
73.2 ± 0.5	
$\mathbf{80.1 \pm 0.6}$	QAP Solver