Parametric Markov Chains: PCTL Complexity and Fraction-free Gaussian Elimination

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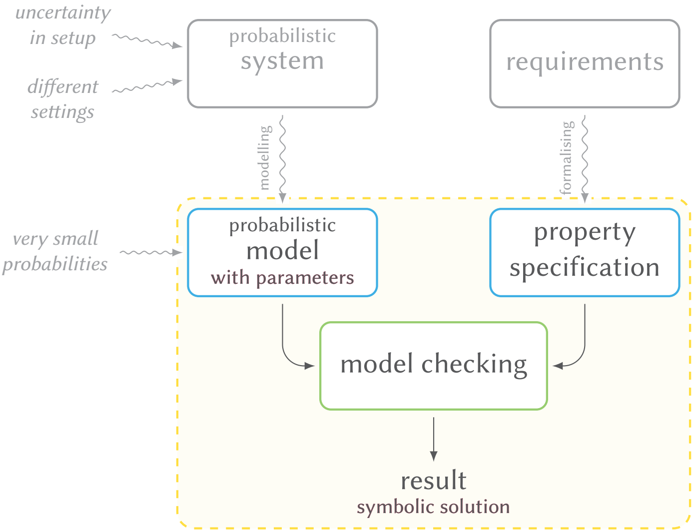
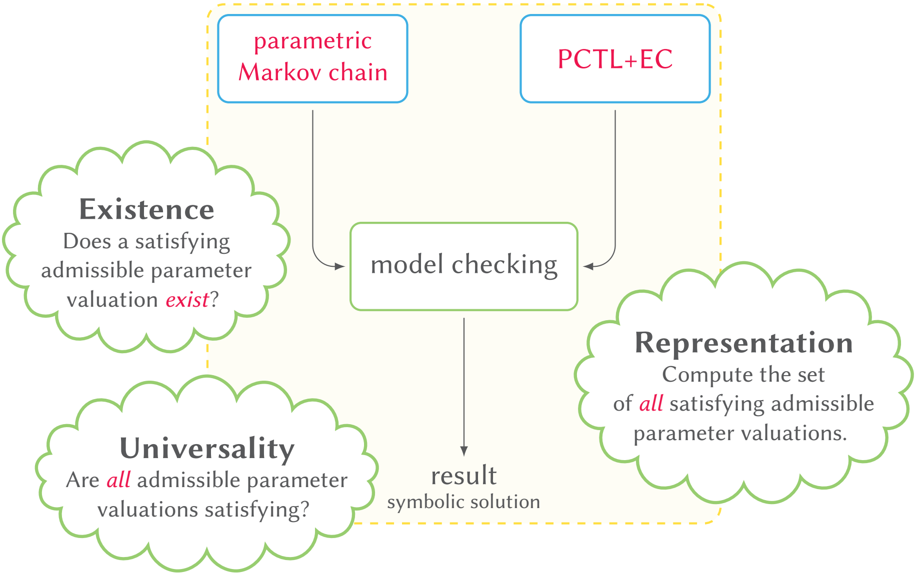
Motivation


uncertainty in setup
different settings
probabilistic system
requirements
modelling
formalising
very small probabilities
probabilistic model with parameters
property specification
model checking
result
symbolic solution

Potential problems



An example

Consider the *Markov chain* $\mathcal{M} = (S, s_{init}, E, P)$:

$$S = \{\textcolor{yellow}{\bullet}, \textcolor{orange}{\bullet}, \textcolor{green}{\bullet}, \textcolor{red}{\bullet}\}$$

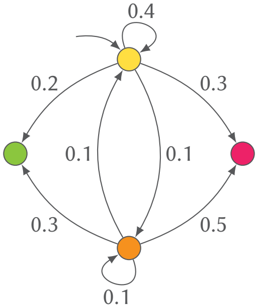
$$s_{init} = \textcolor{yellow}{\bullet} \in S$$

$$E = \{\textcolor{yellow}{\bullet}, \textcolor{orange}{\bullet}\} \times \{\textcolor{yellow}{\bullet}, \textcolor{orange}{\bullet}, \textcolor{green}{\bullet}, \textcolor{red}{\bullet}\} \subseteq S \times S$$

$$P : E \to (0, 1] \quad \text{such that}$$

$$\sum_{t \in S:\ (s,t) \in E} P(s, t) = 1$$

if $s \in S$ not a trap



Probability of reaching $\textcolor{green}{\bullet}$ from $\textcolor{yellow}{\bullet}$:

$$\Pr^{\mathcal{M}}_{\textcolor{yellow}{\bullet}}(\lozenge \textcolor{green}{\bullet}) = 0.2 + 0.4 \cdot 0.2 + 0.1 \cdot 0.3 + 0.4^2 \cdot 0.2 + 0.4 \cdot 0.1 \cdot 0.3 + 0.1^2 \cdot 0.3 + \ldots \quad = \frac{21}{53}$$

An example becomes parametric

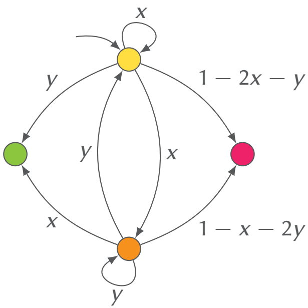
Consider the *parametric* *Markov chain* $\mathfrak{M} = (S, s_{init}, E, \mathbf{P})$:

$S = \{$🟡, 🟠, 🟢, 🔴$\}$

$s_{init} =$ 🟡 $\in S$

$E = \{$🟡, 🟠$\} \times \{$🟡, 🟠, 🟢, 🔴$\} \subseteq S \times S$

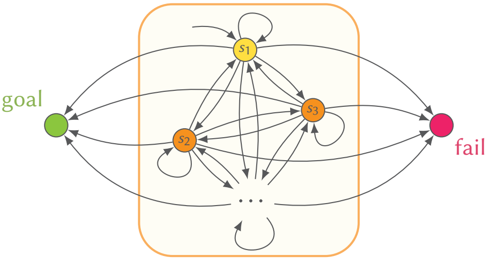
$\mathbf{P} : E \to \mathbb{Q}(x, y)$



Probability of reaching 🟢 from 🟡:

$$\Pr^{\mathfrak{M}}_{🟡}(\lozenge 🟢) = y + x \cdot y + x \cdot x + x^2 \cdot y + x \cdot y \cdot x + x \cdot y^2 + \ldots = \frac{x^2 - y^2 + y}{1 - x - y}$$

Experiment



- *distinct parameters* at each transition not ending in fail:

  $n \cdot (n+1)$ parameters

- transition probabilities towards fail:

  $1 -$ resp. parameters

Time to calculate rational functions, $\Pr^{\mathcal{M}}_{s_i}(\Diamond\, \text{goal})$:

n	parameters	our solver	Storm eigen	Storm state-elim
4	20	0.06 s	0.47 s	0.64 s
5	30	2.13 s	44.47 s	42.09 s
6	42	221.27 s	time-out	time-out

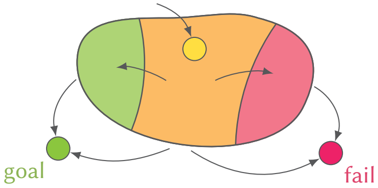
time-out: 30 minutes

Computing reachability probabilities

Problem

- parametric MC $\mathfrak{M} = (S,$ 🟡$, E, \mathbf{P})$
- goal state 🟢 $\in S$
- for all $s \in S$ find

$$p_s = \Pr^{\mathcal{M}}_{s}(\Diamond \text{🟢})$$



Approach

🟩 can reach goal, but not fail: $p_s = 1$

🟥 can reach fail, but not goal: $p_s = 0$

🟧 otherwise:

$$p_s = \sum_{t \in \text{🟧}} \mathbf{P}(s,t) \cdot p_t + \sum_{t \in \text{🟩}} \mathbf{P}(s,t)$$

Gaussian elimination

Standard Gaussian elimination

$$a_{ij}^{(k)} = \frac{a_{ij}^{(k-1)} \cdot a_{k,k}^{(k-1)} - a_{k,j}^{(k-1)} \cdot a_{i,k}^{(k-1)}}{a_{k,k}^{(k-1)}}$$

gcd computations

Division-free Gaussian elimination

$$a_{ij}^{(k)} = a_{ij}^{(k-1)} \cdot a_{k,k}^{(k-1)} - a_{k,j}^{(k-1)} \cdot a_{i,k}^{(k-1)}$$

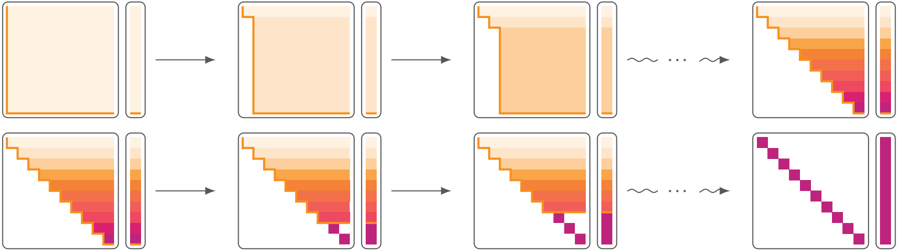
exponent size

One-step fraction-free Gaussian elimination

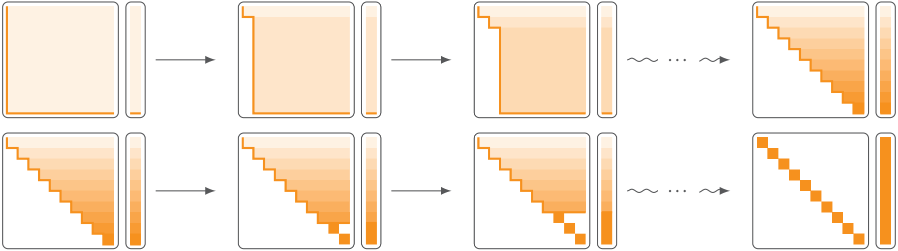
$$a_{ij}^{(k)} = \frac{a_{ij}^{(k-1)} \cdot a_{k,k}^{(k-1)} - a_{k,j}^{(k-1)} \cdot a_{i,k}^{(k-1)}}{a_{k-1,k-1}^{(k-1)}}$$

Gaussian elimination

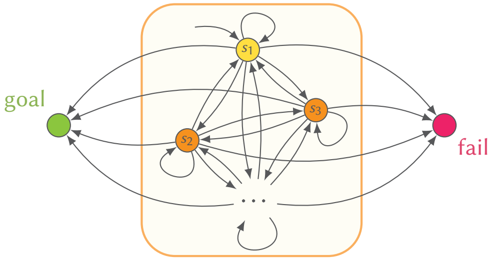
Division-free Gaussian elimination



One-step fraction-free Gaussian elimination



Experiment



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			Storm	
n	parameters	our solver	eigen	state-elim
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Experiments

model	rows	Storm eigen	Storm state-elim	our solver
BRP (128,2), weak-bisim	768	1.386 *s*	1.198 *s*	0.690 *s*
BRP (128,5), weak-bisim	1536	6.938 *s*	9.007 *s*	3.938 *s*
BRP (256,2), weak-bisim	1536	5.922 *s*	4.696 *s*	2.842 *s*
BRP (256,5), weak-bisim	3072	30.632 *s*	37.046 *s*	16.810 *s*
BRP (128,2)	3964	5.519 *s*	6.426 *s*	2.119 *s*
BRP (128,5)	8950	59.814 *s*	112.910 *s*	14.207 *s*
BRP (256,2)	7932	21.827 *s*	27.912 *s*	8.833 *s*
BRP (256,5)	17910	257.712 *s*	499.416 *s*	61.794 *s*
Crowds (3,5), weak-bisim	40	0.077 *s*	0.062 *s*	0.018 *s*
Crowds (5,5), weak-bisim	40	0.076 *s*	0.059 *s*	0.018 *s*
Crowds (10,5), weak-bisim	40	0.077 *s*	0.060 *s*	0.019 *s*
Crowds (3,5)	715	0.989 *s*	0.800 *s*	11.435 *s*
Crowds (5,5)	2928	6.357 *s*	5.505 *s*	1271.954 *s*
Crowds (10,5)	25103	139.818 *s*	173.147 *s*	time-out
Zeroconf (1000)	1002	83.216 *s*	45.239 *s*	50.109 *s*
Zeroconf (10000)	10002	time-out	time-out	time-out

time-out: 30 minutes

Conclusion

Computing probabilities for parametric Markov chains

- drew attention towards *one-step fraction-free* Gaussian elimination
- experiments indicate feasibility of approach
- refinement of implementation still required,
  e. g. structural heuristics, combination with gcd-based approaches

Thank you!