TECHNISCHE UNIVERSITÄT DRESDEN

Fakultät Mathematik und Naturwissenschaften Fachrichtung Mathematik, Institut für Algebra

THE MODEL COMPANIONS OF PERMUTATION PATTERN AVOIDANCE CLASSES

Master-Arbeit
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1 Introduction

Various aspects of pattern avoidance have been studied for the past years with the first related results going as far back as 1915, when MacMahon enumerated the pattern avoidance class Av(123) while studying so-called lattice permutations [16]. The whole subject only became popular in the late 1960s when Knuth published the first volume of his book *The Art of Computer Programming* [15] where he proved that the stack-sortable permutations are precisely those that avoid the pattern 231, and gave an enumeration for this class of permutations. Astonishingly, both the avoidance class Av(123) and the class Av(231) are enumerated by the Catalan numbers.

Since then, many more results have been published regarding enumeration of such pattern avoidance classes. While this is solved for patterns of length 3 [22], the enumeration of Av(1324) is not known to date, and the problem is far from solved for longer patterns.

In 2002 Cameron examined permutation classes from a slightly different angle [7]. Viewing permutations as finite relational structures he proved that apart from the trivial class of permutations there are only five permutation classes which are Fraïssé classes, that is, the age of a countable homogeneous structure.

The scope of this thesis is to analyse some of the model theoretic properties of permutation pattern avoidance classes that are not Fraïssé classes. In particular, it strives to examine in which cases the first-order theory of such a permutation pattern avoidance class has a model companion, and whether the model companion is ω-categorical. If such an ω-categorical model companion exists, then this has some consequences for the avoidance class. Above all else this means that the class has a unique universal limit even if it is not a Fraïssé class. Being able to pinpoint such properties may prove valuable for examining other characteristics of the class, for example when enumerating.

To get a better understanding of the problem at hand the focus in this thesis rests on studying permutation pattern avoidance classes that avoid one or multiple patterns of length 3. While the model theoretic properties will clearly have priority, some enumeration results will also be included if this seems appropriate. Most of them can be found in the publication [22] by Simion and Schmidt which surveys all permutation pattern avoidance classes that avoid patterns of length 3 from a combinatorial point of view.

Chapter 2 is dedicated to all basic notions necessary for the understanding of this thesis. In its first part permutations and the idea of pattern avoidance are introduced including some essential notions and results. The second part gives a very short introduction to model theory chiefly based upon the book *A shorter model theory* by Hodges [13]. The link between model theory and the interpretation of permutations as relational structures will also be established there. This second section also includes several essential model theoretic result regarding ω-categorical theories and model companions.

After this, Chapter 3 then provides new results on the model theoretic classification of several avoidance classes for patterns of length 3. Last but not least, in Chapter 4 an overview of the results of Chapter 3 is given, and the connection to some open problems will be discussed.

2 Essentials

In this chapter the notions essential for the understanding of this thesis are introduced. Furthermore, some already widely known results will be reproduced for reference.

The first section covers definitions and notation relating to permutations, and introduces the idea of patterns and pattern avoidance. For further reading on basic as well as some more advanced notions concerning permutations and pattern avoidance one may refer to Bóna [6].

The second section focuses on a basic introduction to model theory, in particular with regard to the amalgamation property and model completeness. The definitions and notation there largely follow that used by Hodges in [13]. It also includes several well-known results.

2.1 Permutations and Patterns

Before any definitions are given, note that throughout this paper $\mathbb{N}$ refers to the set of natural numbers excluding zero, and whenever natural numbers are mentioned zero is assumed to be excluded. For the few cases where including zero is important, $\mathbb{N}_0$ refers to the natural numbers together with zero, so $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \dots, n\}$. Put more formally,

$$[n] = \{i \in \mathbb{N} \mid i \leq n\}.$$

Aside from natural numbers rational numbers will be needed. Here, $\mathbb{Q}$ will always refer to the set of all rational numbers while $\mathbb{Q}^+$ denotes solely the positive rational numbers.

Next, recall the definition of a total order.

▻ **Definition 2.1 – Total order, strict total order**
Let M be a set. Let $\leq$ be a binary relation on M with the following properties:

- For all $x \in M$ it is $(x, x) \in \leq$. That is, $\leq$ is *reflexive*.
- For all $x, y \in M$, $(x, y) \in \leq$ and $(y, x) \in \leq$ implies $x = y$.
  That is, $\leq$ is *antisymmetric*.
- For all $x, y, z \in M$, $(x, y) \in \leq$ and $(y, z) \in \leq$ implies $(x, z) \in \leq$.
  That is, $\leq$ is *transitive*.
- For all $x, y \in M$ with $x \neq y$ it is either $(x, y) \in \leq$ or $(y, x) \in \leq$.
  That is, $\leq$ is *total*.

Then $\leq$ is called a *total order* on M.

A binary relation $<$ on M that is *irreflexive*, that is, for all $x \in M$, $(x, x) \notin <$, antisymmetric, transitive, and total is called a *strict total order* on M.

Sometimes a total order is also referred to as a linear order. Note also that some authors define total in such a way that it encompasses reflexive, while others would call a relation complete if it is total and reflexive.

▻ **Proposition 2.2**
Let M be a set and $\leq$ a total order on M. Then the binary relation $<$ defined by

$$< := \leq \setminus \{(x, x) \mid x \in M\}$$

is a strict total order. Conversely, let $<'$ be a strict total order on M. Then the binary relation $\leq'$ defined by

$$\leq' := <' \cup \{(x, x) \mid x \in M\}$$

is a total order.

Proof. This follows directly from the definition of total and strict total order. □

In light of this lemma, if a total order is given, the *corresponding* strict total order will always refer to the one defined as above. In general, it will not be stated explicitly. The same applies to a strict total order and the *corresponding* total order defined as above. The symbol $\leq$ will always refer to a total, and the symbol $<$ to a strict total order unless indicated otherwise.

2.1.1 Permutations

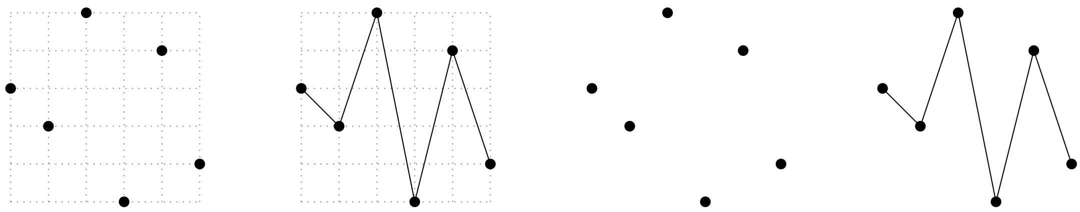
There are many ways in which permutations can be defined. In this case it is useful to regard them as structures carrying two strict total orders.

▻ **Definition – Permutation**
Let $n \in \mathbb{N}$ and let M be a set of n elements. Let $<_1, <_2$ be two strict total orders on M. Then $(M, <_1, <_2)$ is a *permutation* of length n.

This is a very general definition, and to simplify things it is sensible to assume that $M = [n]$, and $<_1$ coincides with the usual $<$-relation on $[n]$. These assumptions are indeed no real restriction as for arbitrary M and $<_1$ a relation $<_2$ and an isomorphism

$$\iota : (M, <_1, <_2) \to ([n], <, <_2)$$

can always be found. That is, among other things, for any $a, b \in M$ it is $a <_1 b$ if and only if $\iota(a) < \iota(b)$. This leads to the following definition which is used throughout the paper if not stated otherwise.



(a) with grid but without connecting lines **(b)** with connecting lines and grid **(c)** with neither grid nor connecting lines **(d)** with connecting lines but without grid

Figure 2.1. Different graphical representations of the permutation $\pi = 436152$. Versions (a) and (d) will be most commonly used in this thesis.

▻ Definition 2.3 – Permutation
Let $n \in \mathbb{N}$. Let $<_1, <_2$ be two total orders on $[n]$ such that $<_1$ coincides with the canonical $<$-relation on $[n]$. Then $([n], <_1, <_2)$ is a *permutation* on $[n]$ of length n.

If a special focus on the length of the permutation is required, a permutation of length n may be referred to as an *n-permutation.* Furthermore, S_n denotes the set of all n-permutations, that is, all permutations on $[n]$. For a given permutation π, the length of π may be denoted as $|\pi|$, and the total orders will be denoted as $<_1^\pi$ and $<_2^\pi$ if omitting the index π would create ambiguity.

For $n \in \mathbb{N}$, id_n denotes the identity permutation of length n, that is, the permutation for which both strict total orders agree on $[n]$. id_0 denotes the empty permutation. The empty permutation will sometimes also be denoted by $\emptyset$.

At times a permutation of length n will be written as a sequence $\pi = (p_1, p_2, \dots, p_n)$, where $\{p_i \mid i \in [n]\} = [n]$, without explicitly mentioning the strict total orders. And if confusion can be ruled out the sequence will be written even shorter as $p_1 p_2 \cdots p_n$ instead. Then, as stated above, the first strict total order is assumed to be the canonical order on $[n]$ while the second is defined implicitly such that $p_1 <_2 p_2 <_2 \cdots <_2 p_n$. That is,

$$<_2 = \left\{(p_i, p_j) \in [n]^2 \mid i < j\right\}.$$

Vice versa, if π is given as as $\pi = ([n], <_1, <_2)$ using strict total orders, then for any $i \in [n]$ it is

$$\begin{aligned} p_i = j, \text{ with } j \in [n] \text{ such that there exist} \\ \text{exactly } i-1 \text{ integers } k \in [n] \text{ with } k <_2 j. \end{aligned}$$

Since $<_2$ is a strict total order this is unambiguous.

For any $i \in [n]$, p_i is the i-th *entry* of the permutation, and i itself is called an *index* of the permutation. For the i-th entry of a permutation π, $\pi(i)$ may be used instead of p_i.

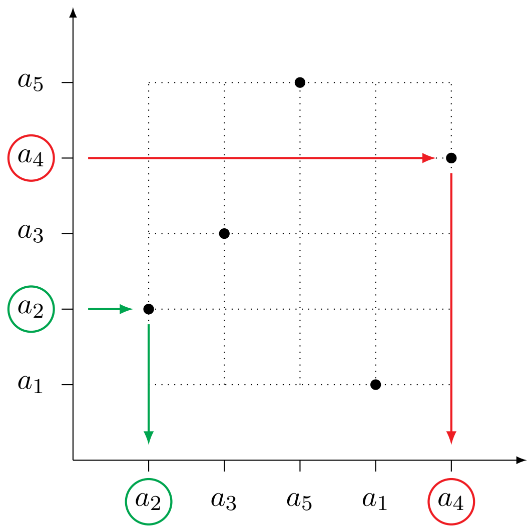


Figure 2.2. Illustration of how to obtain two strict total orders that represent a permutation given graphically. Here $<_1$ was chosen on $\{1, \ldots, 5\}$ such that $a_1 <_1 a_2 <_1 a_3 <_1 a_4 <_1 a_5$. The two sets of coloured arrows show how the positions of a_2 and a_4 in $<_2$ are obtained. The second order $<_2$ can therefore be implicitly stated as $a_2 <_2 a_3 <_2 a_5 <_2 a_1 <_2 a_4$. So if $<_1$ is the canonical order, that is, $a_i = i$, then this yields the permutation $\pi = 23514$.

For example, $([6], <_1, <_2)$ with

$$\begin{aligned} a <_1 b &\text{ if and only if } a < b, \\ a <_2 b &\text{ if and only if } (a, b) \in \{(1,2),(1,5),(3,1),(3,2),(3,5),(3,6), \\ &\qquad (4,1),(4,2),(4,3),(4,5),(4,6), \\ &\qquad (5,2),(6,1),(6,2),(6,5)\}\,, \end{aligned}$$

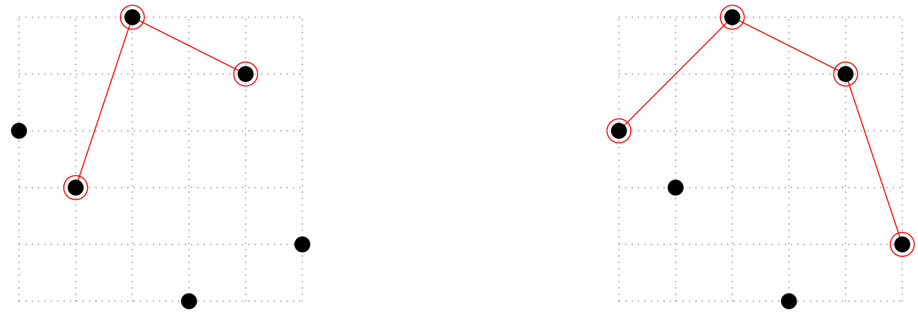
defines the same permutation as $\pi = 436152$.

To further intuition it is useful to represent permutations graphically. This will be done rather intuitively using a grid in the following way.

For an n-permutation π draw a grid with $n \times n$ grid points canonically numbered so that the lower left corner ist $(1, 1)$, and the lower right one is $(n, 1)$, the upper left corner is $(1, n)$, and the upper right one is (n, n). Now all grid points of the form $(i, \pi(i))$ are marked. Sometimes connecting lines between two neighbouring marked grid points with respect to the first component will be included. To simplify the figure the grid can be omitted.

In Figure 2.1 graphical interpretations of the permutation 436152 in line with these principles can be seen. The ones depicted in 2.1 (a) and (d) are most commonly used.

If an n-permutation is given graphically in the described way, two strict total orders representing it can be obtained by choosing the first to be the canonical order on $[n]$, and the



(a) highlighted subsequence 365 isomorphic to the pattern 132

(b) highlighted subsequence 4652 isomorphic to the pattern 2431

Figure 2.3. Permutation $\pi = 436152$ with subsequence isomorphic to pattern 132, and 2431, respectively, highlighted.

second such that if the point at (i, j) was highlighted, $p_i = j$, and $p_1 <_2 p_2 <_2 \ldots <_2 p_n$. More generally speaking, if $<_1$ is chosen such that $a_1 <_1 a_2 <_1 \ldots <_1 a_n$. Then $<_2$ will be of the form $b_1 <_2 b_2 <_2 \ldots <_2 b_n$ with $b_i = a_j$ if (i, j) was marked on the grid.

In the picture this can easily be obtained by assigning the elements of $[n]$ in the order given by $<_1$ to the vertical axis and then finding the second strict total order by traversing from each element to the right to the marked grid point and then writing it down again at that respective position on the horizontal axis as illustrated in Figure 2.2.

2.1.2 Patterns

Patterns are very closely linked to permutations. This becomes very clear with the following definition.

▻ **Definition 2.4 – Pattern, pattern avoidance**
Let $\pi \in S_n$, $\sigma \in S_m$, $m \leq n$. π *contains* σ as a *pattern* if there are indices $i_1, \ldots, i_m$ with $1 \leq i_1 < i_2 < \ldots < i_m < n$ such that for all $j, k \in \{1, \ldots, m\}$ it is $\pi(i_j) < \pi(i_k)$ if and only if $\sigma(j) < \sigma(k)$. This will be denoted by $\sigma \preceq \pi$.

The subsequence $p_{i_1} \cdots p_{i_m}$ of π is called *isomorphic* to σ if the above property holds. This will be denoted by $p_{i_1} \cdots p_{i_m} \cong \sigma$.

Conversely, π is said to *avoid* the pattern σ if there is no such set of indices, that is if π does not contain σ as a pattern. Or in other words, no subsequence of π is isomorphic to σ.

For example the permutation $\pi = 436152$, which has already been illustrated before, contains the pattern 132 as this pattern is isomorphic to the subsequence 365. It also contains the pattern 2431. Figure 2.3 illustrates this nicely. On the other hand, π avoids the pattern 123 since it has no increasing subsequence of length 3.

▻ Definition 2.5 – Permutation pattern avoidance class
Let $\mathcal{F}$ be a set of permutations. Then a permutation π is said to *avoid* $\mathcal{F}$ if for all $\sigma \in \mathcal{F}$, π avoids σ. The permutations in $\mathcal{F}$ are called *forbidden permutations.*

The set of all permutations that avoid $\mathcal{F}$ is denoted by $\mathrm{Av}(\mathcal{F})$ and called *permutation pattern avoidance class*, or *avoidance class* for short. The set of all n-permutations that avoid $\mathcal{F}$ is denoted by $\mathrm{Av}_n(\mathcal{F})$, and $|\mathrm{Av}_n(\mathcal{F})|$ is the number of n-permutations that avoid $\mathcal{F}$.

At times, the elements of $\mathcal{F}$ may be referred to as patterns. In the case that $\mathcal{F}$ consists of just a single or very few permutations, the braces may be omitted when explicitly stating the forbidden permutations. For example, if $\mathcal{F} = \{132, 321\}$ one may write $\mathrm{Av}(132, 321)$ instead of $\mathrm{Av}(\{132, 321\})$.

A very important notion concerning permutation pattern avoidance classes is the following. It was introduced by Atkinson, Murphy and Ruškuc in [2].

▻ Definition 2.6 – Atomic
A permutation pattern avoidance class is called *atomic* if it cannot be defined as a union of two disjoint closed subsets.

The permutation pattern avoidance class $\mathrm{Av}(321, 2143)$ is an example given in [2] for a non-atomic class. It can be decomposed into the two disjoint permutation pattern avoidance classes $\mathrm{Av}(321, 2143, 3142)$ and $\mathrm{Av}(321, 2143, 2413)$.

The following characterisation of atomic permutation pattern classes corresponds to an extract of Theorem 1.2 in [2], and will be restated here without proof.

▻ Theorem 2.7
Let X be a permutation pattern avoidance class. Then the following statements are equivalent.

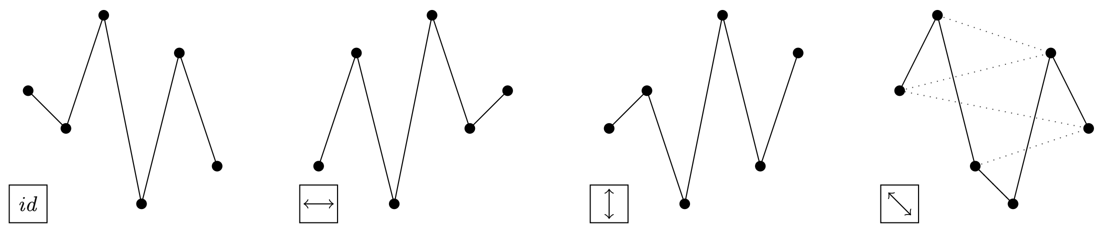
1. X is atomic.
2. For any two permutations π and φ in X there exists $\vartheta \in X$ such that $\pi \preceq \vartheta$ and $\varphi \preceq \vartheta$.

As mentioned earlier, a lot of research has been undertaken enumerating avoidance classes, that is, finding $|\mathrm{Av}_n(\mathcal{F})|$ for all $n \in \mathbb{N}$. In connection with this the following definition has been important. It is being used to combintorially classify avoidance classes.

▻ Definition 2.8 – Wilf-equivalence
Let $\mathcal{F}_1, \mathcal{F}_2$ be two sets of permutations. Then $\mathrm{Av}(\mathcal{F}_1)$ and $\mathrm{Av}(\mathcal{F}_2)$ are said to be *Wilf-equivalent* if

$$|\mathrm{Av}_n(\mathcal{F}_1)| = |\mathrm{Av}_n(\mathcal{F}_2)|$$

for all $n \in \mathbb{N}$.



(a) $\pi = 436152$ **(b)** $\pi^r = 251634$ **(c)** $\pi^c = 341625$ **(d)** $\pi^{-1} = 462153$

Figure 2.4. Original permutation as well as reversal, complement and inverse, using the example of $\pi = 436152$.

2.1.3 Operations on Permutations

▻ **Definition 2.9 – Reversal, complement, inverse**
Let $\pi = p_1 p_2 \cdots p_n$ be an n-permutation. Then the permutation π^r defined by reversing the order of the entries of π,

$$\pi^r = p_n \cdots p_2 p_1,$$

is called the *reversal* of π. Similarly, the permutation π^c gained by replacing each entry p_i with $n + 1 - p_i$,

$$\pi^c = (n + 1 - p_1, n + 1 - p_2, \ldots, n + 1 - p_n)\,,$$

is called the *complement* of π. The *inverse* of π is the permutation π^{-1} defined in such a way that for all $i \in [n]$, $\pi^{-1}(i) = j$ if and only if $\pi(j) = i$.

Note that this definition of the inverse of a permutation agrees with that of the group theoretic inverse.

To get a better idea of these operations it is sensible to once again look at a small example. Consider the permutation $\pi = 436152$. Then it is easy to see that

$$\pi^r = 251634, \qquad \pi^c = 341625, \qquad \pi^{-1} = 462153.$$

An illustration of this can be found in Figure 2.4. A closer look at it suggests that it is sensible to interpret these three operations geometrically. In the example, π^r is obtained from π by flipping it along a vertical axis, while π^c is obtained by flipping along a horizontal axis. For the inverse it is not so easy to see in the chosen representation, but it is obtained by flipping along a diagonal axis with slope 1. The small pictograms in the figure indicate these transformations.

This is of course not a coincidence. It can easily be seen from the definition that for any permutation π the reverse, complement, and inverse can be obtained geometrically by apply-

Table 2.1. Illustrated Cayley table for symmetry group of a square, D_4, with small indicators for the reverse, complement, and inverse of a permutation.

	id	90	180	270	↔ r	↕ c	↗	↘ -1
id	*id*	90	180	270	↔	↕	↗	↘
90	90	180	270	*id*	↗	↘	↕	↔
180	180	270	*id*	90	↕	↔	↘	↗
270	270	*id*	90	180	↘	↗	↔	↕
↔ r	↔	↘	↕	↗	*id*	180	270	90
↕ c	↕	↗	↔	↘	180	*id*	90	270
↗	↗	↔	↘	↕	90	270	*id*	180
↘ -1	↘	↕	↗	↔	270	90	180	*id*

ing these transformations, that is, by flipping vertically, horizontally, or along an increasing diagonal axis.

From group theory it is known that these three symmetries form a (non-minimal) generating set of the dihedral group D_4 of a square. It makes sense to look at this group since the geometrical representation of a permutation lives on an $n \times n$-grid, that is, essentially, a square. For reference, Table 2.1 shows the Cayley table of D_4 using pictograms for each transformation. Therefore, for a permutation π the symmetries can be written in the following way. The identity is omitted here, as it yields the permutation itself.

$$\begin{aligned}
\pi^r &\text{ corresponds to a flip along a vertical axis,}\\
\pi^c &\text{ corresponds to a flip along a horizontal axis,}\\
\pi^{-1} &\text{ corresponds to a flip along an increasing diagonal,}\\
((\pi^c)^{-1})^c &\text{ corresponds to a flip along a decreasing diagonal,}\\
(\pi^c)^{-1} &\text{ corresponds to a rotation of 90 degrees,}\\
(\pi^r)^c &\text{ corresponds to a rotation of 180 degrees,}\\
(\pi^{-1})^c &\text{ corresponds to a rotation of 270 degrees.}
\end{aligned}$$

Of course, these are not the only ways in which the respective symmetries can be represented as one can easily see from the Cayley table. From now on, D_4 will also refer to the set of all symmetries that can be applied to a permutation. So for any permutation π and $\tau \in D_4$, $\tau(\pi)$ is the permutation obtained from π by applying τ.

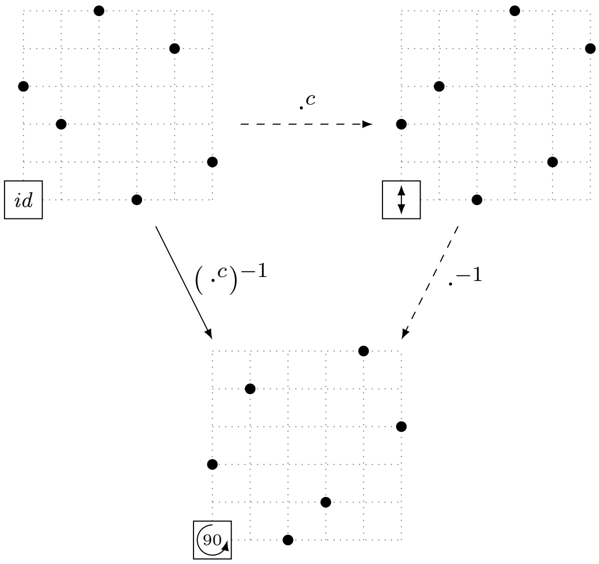


Figure 2.5. Rotation about 90 degrees for $\pi = 436152$.

For example, if $\pi = 436152$ and τ is the rotation of 90 degrees,

$$\tau(\pi) = (\pi^c)^{-1} = (436152^c)^{-1} = 341625^{-1} = 351264.$$

A geometric interpretation can be seen in Figure 2.5. The small pictograms refer to the symmetry with respect to the original permutation.

The following definition formalises a few notions and notations regarding the symmetries of a permutation.

▻ **Definition 2.10 – Symmetric permutation, symmetric avoidance class**
Two permutations π, σ are said to be *symmetric* if there exists a symmetry $\tau \in D_4$ such that $\sigma = \tau(\pi)$.

Let $\mathcal{F}$ be a set of permutations, and $\tau \in D_4$ an arbitrary symmetry. Then $\tau(\mathcal{F})$ denotes the set of symmetric permutations obtained by applying τ to each element of the set, that is,

$$\tau(\mathcal{F}) = \{\tau(\pi) \mid \pi \in \mathcal{F}\}.$$

Two avoidance classes $\text{Av}(\mathcal{F})$ and $\text{Av}(\mathcal{F}')$ are said to be *symmetric* to each other if there exists $\tau \in D_4$ such that $\mathcal{F}' = \tau(\mathcal{F})$. Otherwise, the classes as well as the sets of avoided permutations are called *non-symmetric* to each other.

The following lemma, a special case of which can be found in [22], justifies this notion of symmetric avoidance classes. It will later on be useful to narrow down the number of cases.

▻ **Lemma 2.11**
Let $\mathcal{F}$ be a set of patterns, and $\tau \in D_4$ a symmetry. Then $\tau(\pi) \in \text{Av}(\tau(\mathcal{F}))$ for all $\pi \in \text{Av}(\mathcal{F})$.

Proof. First, note that it suffices to show that for any $\pi \in \mathrm{Av}(\mathcal{F})$ and $\sigma \in \mathcal{F}$, $\tau(\pi)$ avoids $\tau(\sigma)$. Also note that it suffices to show this for a generating set of D_4 instead of all eight transformations. One such set would be that consisting of a reflection about a vertical axis and one about an increasing diagonal, that is, $G = \{ \cdot^r, \cdot^{-1}\}$. Now for these two cases, $\tau = \cdot^r$ and $\tau = \cdot^{-1}$, the above statement will be proven.

Choose $\pi \in \mathrm{Av}(\mathcal{F})$ and $\sigma \in \mathcal{F}$ arbitrarily. Now assume that $\pi' = \tau(\pi)$ does not avoid $\sigma' = \tau(\sigma)$. Obviously, applying a transformation does not change the length of the respective permutation. So it is possible to write $|\pi| = |\pi'| = n$, and $|\sigma| = |\sigma'| = m$. It can be assumed that $m \leq n$ since a permutation always avoids any longer permutations.

Therefore, there is a subsequence $p'_{i_1}p'_{i_2}\cdots p'_{i_m}$ of π' isomorphic to $\sigma' = s'_1 \cdots s'_m$.

Case 1: $\tau = \cdot^r$. It is known that $\cdot^r$ is its own inverse in D_4. Hence, $\pi'^r = \pi$ and $\sigma'^r = \sigma$. Focusing on the entries $p'_{i_1}, p'_{i_2}, \ldots, p'_{i_m}$ of π' it is for any $k \in [m]$

$$p_{n+1-i_k} = p'_{i_k}, \qquad\qquad s_{m+1-k} = s'_k.$$

As $p'_{i_1}p'_{i_2}\cdots p'_{i_m}$ is isomorphic to σ', it is for all $j, k \in [m]$

$$\begin{aligned} & p'_{i_k} < p'_{i_j} \text{ if and only if } s'_k < s'_j \\ \Rightarrow \quad & p_{n+1-i_k} < p_{n+1-i_j} \text{ if and only if } s_{m+1-k} < s_{m+1-j}, \end{aligned}$$

where $n+1-i_k$ and $n+1-i_j$ are ordered in the same way as $m+1-k$ and $m+1-j$. Since k, j can be any natural number from 1 to m, this implies that π contains σ. To be precise, the subsequence $p_{n+1-i_m}p_{n+1-i_{m-1}}\cdots p_{n+1-i_1}$ is isomorphic to

$$\sigma = s_{m+1-m}s_{m+1-m+1}\cdots s_{m+1-1} = s_1s_2\cdots s_m.$$

This contradicts the initial premise that π avoids σ. Therefore, the assumption does not hold in this case.

Case 2: $\tau = \cdot^{-1}$. Like $\cdot^r$ above, $\cdot^{-1}$ is its own inverse in D_4. Therefore, $\pi'^{-1} = \pi$ and $\sigma'^{-1} = \sigma$. Again, focusing on the entries $p'_{i_1}, p'_{i_2}, \ldots, p'_{i_m}$ of π' yields for any $k \in [m]$

$$p_{p'_{i_k}} = i_k, \qquad\qquad s_{s'_k} = k.$$

Since the sequences $(i_k)_{k=1}^m$ and $(k)_{k=1}^m$ are order isomorphic, and so are the sequences $(p'_{i_k})_{k=1}^m$ and $(s'_k)_{k=1}^m$, this means that the sequences $(p_{p'_{i_k}})_{k=1}^m$ and $(s_{s'_k})_{k=1}^m$ are order isomorphic as well. Therefore, π contains σ since the first sequence is a subsequence of π with m entries and the second is precisely σ. Thus, the above assumption leads to a contradiction.

Hence, the assumption cannot hold in either case. Therefore, if π avoids σ then $\tau(\pi)$ avoids $\tau(\sigma)$ as well. This finally yields the stated assertion. □

▻ Corollary 2.12
Let π, σ be permutations, and let $\tau \in D_4$ be a symmetry. Then π contains σ as a pattern if and only if $\tau(\pi)$ contains $\tau(\sigma)$ as a pattern.

Proof. This follows directly from Lemma 2.11. Assuming that the statement does not hold results in a contradiction due to said lemma. □

2.1.4 Sums of permutations

Another interesting notion is that of combining two permutations by summing them up. Throughout this paper two kinds of sums, introduced in the following definition, will be relevant.

▷ **Definition 2.13 – Direct sum, skew sum**
Let $n, m \in \mathbb{N}$. Let π be a n-permutation, and φ an m-permutation. Then the *direct sum* of π and φ, $\pi \oplus \varphi$, is defined as the $(m+n)$-permutation with

$$(\pi \oplus \varphi)(i) = \begin{cases} \pi(i), & \text{if } i \in [n] \\ \varphi(i-n) + m, & \text{if } i \in [m+n] \setminus [n] \end{cases}.$$

On the other hand, the $m + n$-permutation $\pi \ominus \varphi$ defined by

$$(\pi \ominus \varphi)(i) = \begin{cases} \pi(i) + m, & \text{if } i \in [n] \\ \varphi(i-n), & \text{if } i \in [m+n] \setminus [n] \end{cases}$$

is called the *skew sum* of π and φ.

Note the following convention. If any of the permutations is empty, that is, of length 0, the equations

$$\emptyset \oplus \varphi = \varphi \qquad \emptyset \ominus \varphi = \varphi$$
$$\pi \oplus \emptyset = \pi \qquad \pi \ominus \emptyset = \pi$$
$$\emptyset \oplus \emptyset = \emptyset \qquad \emptyset \ominus \emptyset = \emptyset$$

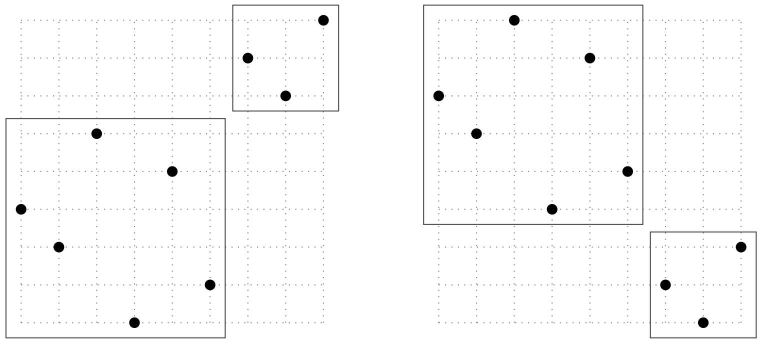
define the respective $\ominus$- and $\oplus$-sums.

To illustrate the direct and the skew sum, look at the following example. Let $\pi = 436152$ and $\varphi = 213$. Then

$$\pi \oplus \varphi = 436152879,$$
$$\pi \ominus \varphi = 769485213.$$

For both sums the first six entries are isomorphic to π, and the last three to φ.

The geometric analogy for both sums is obtained by looking at an $(n+m) \times (n+m)$ grid. Drawing the first permutation on the $n \times n$-subgrid starting in the lower left corner, and the second on the $m \times m$-subgrid ending in the upper right corner, yields the direct sum of two permutations. Drawing the first permutation on the $n \times n$-subgrid containing the upper left corner, and the second on the $m \times m$-subgrid that contains the the lower right corner, yields the respective skew sum. Figure 2.6 demonstrates this for the example above.



(a) $\pi \oplus \varphi = 436152879$ **(b)** $\pi \ominus \varphi = 769485213$

Figure 2.6. Direct sum and skew sum of $\pi = 436152$ and $\varphi = 213$.

▻ **Definition 2.14 – Decomposable, indecomposable, sum-complete**
A permutation π is said to be $\oplus$-*decomposable*, or *decomposable*, if there exist permutations φ and ϑ of at least length 1 such that $\pi = \varphi \oplus \vartheta$. Otherwise, π is $\oplus$-*indecomposable*, or *indecomposable* for short.

Analogously, if there are such permutations so that $\pi = \varphi \ominus \vartheta$ then π is called $\ominus$-*decomposable*. If not, then π is said to be $\ominus$-*indecomposable*.

A class of permutations is called $\oplus$-*sum-complete*, usually shortened to *sum-complete*, if the direct sum of any two permutations in that class is also in it. Analogously, it is $\ominus$-*sum-complete* if the skew sum of any two elements is in the class as well.

Obviously, any sum-complete avoidance class is atomic. Otherwise, one could just consider a permutation from each of the two disjoint subsets. Then the direct sum of these has to be in one of the subsets. Since it contains both permutations as a pattern, this yields a contradiction with the initial assumption that the two subsets are closed. The following sums this up in a lemma.

▻ **Lemma 2.15**
Any sum-complete permutation pattern avoidance class is atomic.

Not every atomic set is sum-complete, though. For example, Av(132) is atomic by Theorem 2.7 as the skew sum of any two permutations in the class is in it as well. But for $1 \in \text{Av}(132)$ and $21 \in \text{Av}(132)$ it is $1 \oplus 21 = 132 \notin \text{Av}(132)$. So the class is not sum-complete.

▻ **Corollary 2.16**
Any $\ominus$-sum-complete permutation pattern avoidance class is atomic.

Proof. This follows directly from Lemma 2.15 in conjunction with Lemma 2.11. □

The following lemma from [2] and the associated corollary obtained by applying Lemma 2.11 help to easily identify sum-complete and $\ominus$-sum-complete avoidance classes. The proof will be omitted here.

▻ **Lemma 2.17**
Let $\mathcal{F}$ be a set of permutations. Then $\mathrm{Av}(\mathcal{F})$ is sum-complete if and only if all permutations in $\mathcal{F}$ are indecomposable.

▻ **Corollary 2.18**
Let $\mathcal{F}$ be a set of permutations. Then $\mathrm{Av}(\mathcal{F})$ is $\ominus$-sum-complete if and only if all permutations in $\mathcal{F}$ are $\ominus$-indecomposable.

2.2 Model theory

This section is going to provide a short introduction to model theory as well as some results important to the work in this thesis. As mentioned earlier, the following definitions and theorems were mainly taken from [13], and more information on them as well as a more detailed discussion can be found there.

2.2.1 Relational Structures and some Properties

The following definitions introduce some very basic notions of model theory. As this thesis is primarily about permutation pattern avoidance classes the link to permutations will now be established as well.

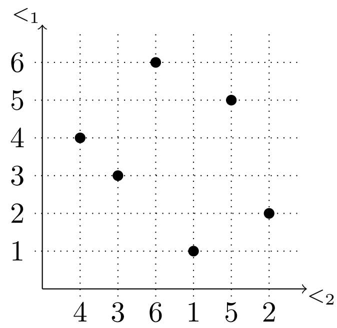
▻ **Definition 2.19 – Signature, structure, relational structure**
A *signature* is a tuple $L = (F, R, C, \mathrm{ar})$ where

- F the set of so-called *function symbols*, and
- R the set of so-called *relation symbols*,
- C is the set of so-called *constant symbols*.

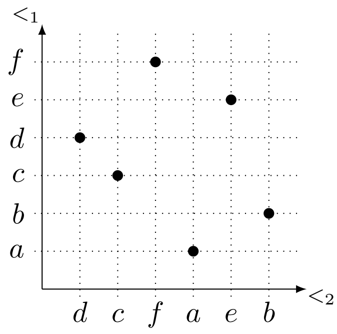
The symbols in C, F, and R are assumed to be disjoint. ar is a function determining the arity of the function and relational symbols, that is, $\mathrm{ar} : F \cup R \to \mathbb{N}$.

An *L-structure* $\mathcal{A}$ is a tuple $(A, (f^{\mathcal{A}})_{f\in F}, (\rho^{\mathcal{A}})_{\rho\in R}, (c^{\mathcal{A}})_{c\in C})$, where

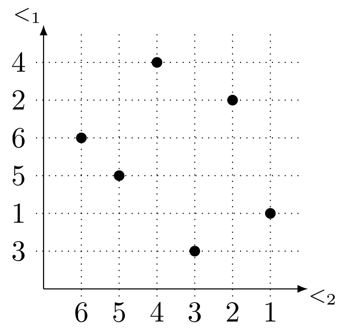
- A is a set,
- $f^{\mathcal{A}} : A^{\mathrm{ar}(f)} \to A$ for all $f \in F$, that is, each $f^{\mathcal{A}}$ is the interpretation of the respective function symbol in $\mathcal{A}$,
- $\rho^{\mathcal{A}} \subseteq A^{\mathrm{ar}(\rho)}$ for all $\rho \in R$, that is, each $\rho^{\mathcal{A}}$ is the interpretation of the respective relation symbol in $\mathcal{A}$,
- $c^{\mathcal{A}} \in A$ for all $c \in C$, that is, each $c^{\mathcal{A}}$ is the interpretation of the respective constant symbol in $\mathcal{A}$.



(a) Permutation $\pi = 436152$ with canonical strict total orders.



(b) $\{<_1, <_2\}$-structure $\mathcal{P}$ with domain $\{a, b, c, d, e, f\}$.



(c) $\{<_1, <_2\}$-structure $\mathcal{Q}$ with domain $\{1, 2, 3, 4, 5, 6\}$.

Figure 2.7. Graphical representation of the permutation $\pi = 436152$, and two more $\{<_1, <_2\}$-structures representing the same permutation.

The set A is called the *domain* of $\mathcal{A}$, and sometimes denoted as $\operatorname{dom}(\mathcal{A})$.

A signature L where F and C are empty is called a *relational signature.* A structure with a relational signature is called a *relational structure.*

Note that from now on it will always be assumed that all structures are relational structures. Instead of denoting the signature as a tuple $L = (R, \mathrm{ar})$ it will usually be written simply as $L = R$ with the arities being mentioned separately. The following definitions will be given with regard to relational structures only. If applicable a more general form can be found in [13].

Note also that more often than not the interpretation $\rho^{\mathcal{A}}$ of a specific relation in $\mathcal{A}$ will just be written as ρ. That is, as long as the associated structure is obvious from the context.

From Definition 2.3 it is easy to see that an n-permutation can also be regarded as a relational structure with a finite domain with n elements and a relational signature $\{<_1, <_2\}$ where $<_1$ and $<_2$ are binary relation symbols, and their interpretations are strict total orders on the domain.

Obviously, there is no unique representation of a permutation in this way. For example, the $\{<_1, <_2\}$-structures $\mathcal{P} = (\{a, b, c, d, e, f\}, <_1^{\mathcal{P}}, <_2^{\mathcal{P}})$ and $\mathcal{Q} = (\{1, 2, 3, 4, 5, 6\}, <_1^{\mathcal{Q}}, <_2^{\mathcal{Q}})$ with $<_1, <_2$ interpreted as strict total orders given implicitly by

$$
\begin{aligned}
&a <_1^{\mathcal{P}} b <_1^{\mathcal{P}} c <_1^{\mathcal{P}} d <_1^{\mathcal{P}} e <_1^{\mathcal{P}} f \qquad && 3 <_1^{\mathcal{Q}} 1 <_1^{\mathcal{Q}} 5 <_1^{\mathcal{Q}} 6 <_1^{\mathcal{Q}} 2 <_1^{\mathcal{Q}} 4 \\
&d <_2^{\mathcal{P}} f <_2^{\mathcal{P}} b <_2^{\mathcal{P}} a <_2^{\mathcal{P}} e <_2^{\mathcal{P}} c && 6 <_2^{\mathcal{Q}} 5 <_2^{\mathcal{Q}} 4 <_2^{\mathcal{Q}} 3 <_2^{\mathcal{Q}} 2 <_2^{\mathcal{Q}} 1
\end{aligned}
$$

both correspond to the permutation $\pi = 436152$, whose canonical representation was given by

$$1 <_1 2 <_1 3 <_1 4 <_1 5 <_1 6$$
$$4 <_2 3 <_2 6 <_2 1 <_2 5 <_2 2$$

in the associated example in Section 2.1. Figure 2.7 illustrates this neatly.

▻ **Definition 2.20 – Homomorphism, embedding, isomorphism, automorphism**
Let $\mathcal{A}$ and $\mathcal{B}$ be two L-structures with $L = (R, \mathrm{ar})$. A *homomorphism* h from $\mathcal{A}$ to $\mathcal{B}$, denoted as $h : \mathcal{A} \to \mathcal{B}$, is a function $h : \mathrm{dom}\,(\mathcal{A}) \to \mathrm{dom}\,(\mathcal{B})$ such that for all $\rho \in R$ and $a = (a_1, \dots, a_{\mathrm{ar}(\rho)}) \in \mathrm{dom}\,(\mathcal{A})^{\mathrm{ar}(\rho)}$,

$$a \in \rho^{\mathcal{A}} \Rightarrow h(a) := (h(a_1), \dots h(a_{\mathrm{ar}(\rho)})) \in \rho^{\mathcal{B}}.$$

An *embedding* is an injective homomorphism $h : \mathcal{A} \to \mathcal{B}$ with

$$\forall \rho \in R \,\forall a \in \mathrm{dom}\,(\mathcal{A})^{\mathrm{ar}(\rho)} : a \in \rho^{\mathcal{A}} \Leftrightarrow h(a) \in \rho^{\mathcal{B}}.$$

A surjective embedding is called an *isomorphism*, and if there exists an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ then $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic*, denoted by $\mathcal{A} \cong \mathcal{B}$. An isomorphism from $\mathcal{A}$ to itself is called an *automorphism*.

From the definition of isomorphic structures it can be seen that two $\{<_1, <_2\}$-structures represent the same permutation in the sense of Definition 2.3 if they are isomorphic. So it makes sense to take a closer look at so-called *isomorphism types* which represent structures isomorphic to each other.

This way, each permutation corresponds to a unique isomorphism type. Obviously, two isomorphic $\{<_1, <_2\}$-structures where $<_1, <_2$ are strict total orders have the same graphical interpretation when the axes are not labelled. Therefore, it makes sense to omit the labelling when representing permutations graphically as this sufficiently defines the isomorphism type. This way all three permutations in the example illustrated by Figure 2.7 would look the same as they represent the same isomorphism type.

▻ **Definition 2.21 – Substructure**
If $\mathcal{A}$ and $\mathcal{B}$ are L-structures with $\mathrm{dom}\,(\mathcal{A}) \subseteq \mathrm{dom}\,(\mathcal{B})$ for which the inclusion map

$$\iota : \mathrm{dom}\,(\mathcal{A}) \to \mathrm{dom}\,(\mathcal{B}) : a \mapsto a$$

is an embedding, then $\mathcal{A}$ is called a *substructure* of $\mathcal{B}$, and $\mathcal{B}$ an *extension* of $\mathcal{A}$. This will be denoted by $\mathcal{A} \leq \mathcal{B}$.

Note that for a relational structure $\mathcal{A}$ with signature $L = (R, \mathrm{ar})$, and any set $B \subseteq \mathrm{dom}\,(\mathcal{A})$, the L-structure defined by $\mathcal{B} := (B, (\rho^{\mathcal{B}})_{\rho \in R})$ with

$$\rho^{\mathcal{B}} = \rho^{\mathcal{A}} \cap B^{\mathrm{ar}(\rho)}$$

for all $\rho \in R$, is a substructure of $\mathcal{A}$. $\mathcal{B}$ is said to be *generated by* B. If B if finite, then $\mathcal{B}$ is said to be *finitely generated*. Note also that any finitely generated relational structure is finite. Therefore, it is possible to interchangeably use the terms finitely generated structure and finite structure if the signature is relational.

Looking back at Definition 2.4 it is obvious that if permutations are seen as $\{<_1, <_2\}$-structures as explained above then a permutation π *contains another permutation* σ *as a pattern* if π has a substructure that is isomorphic to σ, or in other words, σ can be embedded into π. Conversely, π *avoids* σ if none of its substructures are isomorphic to σ. So a permutation pattern avoidance class can be regarded as a class of relational structures that avoid certain structures as substructures.

▻ Definition 2.22 – Age, universal
Let L be a relational signature, and $\mathcal{A}$ an L-structure. The *age* of $\mathcal{A}$, denoted by $\operatorname{Age}(\mathcal{A})$ is the class $\mathscr{A}$ of all finitely generated structures that can be embedded into $\mathcal{A}$. $\operatorname{Age}(\mathcal{A})$ is said to be countable if the class contains only countably many isomorphism types. A class $\mathscr{A}$ of L-structures will be called an *age* if it is the age of some L-structure.

A countable structure $\mathcal{A}$ of age $\mathscr{A}$ is said to be *universal* for $\mathscr{A}$, or simply *universal*, if every finite or countable structure $\mathcal{B}$ whose age is a subset of $\mathscr{A}$ is embeddable in $\mathcal{A}$, or in other words, if

$$\operatorname{Age}(\mathcal{B}) \subseteq \operatorname{Age}(\mathcal{A}),$$

then there exists an embedding $\iota : \mathcal{B} \to \mathcal{A}$.

▻ Definition 2.23 – Isomorphism closed, hereditary property
Let L be a relational signature, and $\mathscr{X}$ a class of L-structures. Then the following define two properties such a class may have.

- $\mathscr{X}$ is *isomorphism closed* if for any L-structure $\mathcal{A} \in \mathscr{X}$, and L-structure $\mathcal{B}$, $\mathcal{A} \cong \mathcal{B}$ implies that $\mathcal{B} \in \mathscr{X}$ as well.
- $\mathscr{X}$ has the *hereditary property* if for any $\mathcal{A} \in \mathscr{X}$, and $\mathcal{B} \leq \mathcal{A}$ where $\mathcal{B}$ is finitely generated, it also holds that $\mathcal{B} \in \mathscr{X}$.

The following lemma, and in particular its corollary, connect these two very basic properties to permutation pattern avoidance classes. Albeit its very basic nature it is essential for the results stated later on.

▻ Lemma 2.24
Let L be a relational signature, and $\mathscr{F}$ a finite set of L-structures. Let $\mathscr{X}$ be the class of all relational L-structures that avoid the set $\mathscr{F}$, that is, do not have a substructure isomorphic to a structure in $\mathscr{F}$. Then $\mathscr{X}$ is isomorphism closed, and has the hereditary property.

Proof. Suppose that $\mathscr{X}$ is not isomorphism closed. Then there exists an L-structure $\mathcal{A} \in \mathscr{X}$ and an L-structure $\mathcal{B}$ with $\mathcal{A} \cong \mathcal{B}$, but $\mathcal{B} \notin \mathscr{X}$. Thus, there exists $\mathcal{F} \in \mathscr{F}$, and an L-structure $\mathcal{C}$ such that $\mathcal{F} \cong \mathcal{C}$ and $\mathcal{C} \leq \mathcal{B}$.

Let $\iota : \mathcal{B} \to \mathcal{A}$ denote the isomorphism between $\mathcal{A}$ and $\mathcal{B}$. Since $\mathcal{C}$ is a substructure of $\mathcal{B}$ it is $\operatorname{dom}(C) \subseteq \operatorname{dom}(B)$. Now consider the substructure $\mathcal{D}$ of $\mathcal{A}$ generated in $\mathcal{A}$ by the image $\iota(\operatorname{dom}(\mathcal{C}))$. Because of ι being an isomorphism, $\mathcal{D} \cong \mathcal{C}$. Hence, $\mathcal{D} \cong \mathcal{F}$. This implies that $\mathcal{A}$ has a substructure isomorphic to a structure in $\mathscr{F}$.

Hence, $\mathcal{A}$ cannot be in $\mathscr{X}$. This contradicts the assumptions. Therefore, $\mathscr{X}$ is closed under isomorphism.

Suppose now that $\mathscr{X}$ does not have the hereditary property. Thus, a structure $\mathcal{R} \in \mathscr{X}$ exists that has a finitely generated substructure $\mathcal{S} \notin \mathscr{X}$. Then there exists $\mathcal{E} \in \mathscr{F}$ isomorphic to a substructure $\mathcal{T}$ of $\mathcal{S}$.

Thus, $\mathcal{T} \leq \mathcal{S} \leq \mathcal{R}$. Then $\mathcal{T} \leq \mathcal{R}$. So $\mathcal{R}$ has a substructure isomorphic to $\mathcal{E}$, and cannot be in $\mathscr{X}$. Since this falsifies the assumptions, $\mathscr{X}$ does have the hereditary property. □

▹ Corollary 2.25
Any permutation pattern avoidance class is closed under isomorphism, and has the hereditary property.

Proof. For any permutation pattern avoidance class $\operatorname{Av}(\mathcal{F})$ that avoids $\mathcal{F}$ this follows directly from the previous lemma with $\mathscr{X}$ being the class of $\{<_1, <_2\}$-structures isomorphic to the permutations in $\operatorname{Av}(\mathcal{F})$ and $\mathscr{F}$ a set of $\{<_1, <_2\}$-structures isomorphic to the permutations in $\mathcal{F}$. □

The next property is one that not all permutation pattern avoidance classes share. It corresponds to the notion of atomic defined in Section 2.1.2.

▹ Definition 2.26 – Joint embedding property
Let L be a relational signature and $\mathscr{X}$ a class of L-structures. Then $\mathscr{X}$ has the *joint embedding property* if for all $\mathcal{A}, \mathcal{B} \in \mathscr{X}$ there exists $\mathcal{C} \in \mathscr{X}$ such that $\mathcal{A}$ and $\mathcal{B}$ can be embedded into $\mathcal{C}$, that is, there exist embeddings $\iota_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}$ and $\iota_{\mathcal{B}} : \mathcal{B} \to \mathcal{C}$.

The fact that the joint embedding property holds for a permutation pattern avoidance class if and only if the class is atomic can be deduced directly from Theorem 2.7, since the interpretation of embeddable substructures as patterns contained in a permutation agrees with the second characterisation in said theorem.

The following theorem is due to Fraïssé, and can also be found as Theorem 6.1.1 in [13]. The proof will be omitted here.

▹ Theorem 2.27
Let L be a relational signature, and $\mathscr{X}$ a non-empty class of L-structures that contains only finitely or countably many isomorphism types.

Then $\mathscr{X}$ is the age of a finite or countable L-structure if and only if $\mathscr{X}$ is isomorphism closed and has the hereditary and joint embedding property.

By this it is known that, in particular, a permutation pattern avoidance class is atomic if and only if it is the age of some finite or countable $\{<_1, <_2\}$-structure.

2.2.2 The Amalgamation Property

An even more specialised property that some classes may have is introduced in the following definition.

▶ **Definition 2.28 – Amalgamation property**
Let L be a relational signature and $\mathscr{X}$ a class of L-structures. Then $\mathscr{X}$ has the *amalgamation property* if for all $\mathcal{A}, \mathcal{B}_1, \mathcal{B}_2 \in \mathscr{X}$ with embeddings $\iota_1 : \mathcal{A} \to \mathcal{B}_1$ and $\iota_2 : \mathcal{A} \to \mathcal{B}_2$ there exists $\mathcal{C} \in \mathscr{X}$ and embeddings $\eta_1 : \mathcal{B}_1 \to \mathcal{C}$, $\eta_2 : \mathcal{B}_2 \to \mathcal{C}$ such that $\eta_1 \iota_1 = \eta_2 \iota 2$.

▶ **Definition 2.29 – Homogeneous**
A relational L-structure $\mathcal{A}$ is called *homogeneous* if any isomorphism between finitely generated substructures extends to an automorphism on $\mathcal{A}$.

Some authors, among them Hodges [13], use the term *ultrahomogeneous* for this notion instead in order to prevent confusion with other properties that are called homogeneous as well. As this is not going to be a problem here, the definition given above will be used like this throughout the thesis.

The following theorem due to Fraïssé will be stated without proof. A detailed proof can be found in [13] among others, where this theorem is presented as Theorem 6.1.2.

▶ **Theorem 2.30 – Fraïssé's Theorem**
Let L be a countable relational signature and $\mathscr{X}$ a non-empty finite or countable class of finitely generated L-structures. If $\mathscr{X}$ is closed under isomorphism, and has the hereditary property, joint embedding property, and amalgamation property, then there exists a countable L-structure $\mathcal{A}$, unique up to isomorphism, such that $\mathcal{A}$ is homogeneous and $\text{Age}(\mathcal{A}) = \mathscr{X}$.

If $\mathcal{A}$ exists, it is called the *Fraïssé limit* of $\mathscr{X}$. If $\mathscr{X}$ has a Fraïssé limit, $\mathscr{X}$ is called a *Fraïssé class*, or *amalgamation class.*

It was noted by Cherlin [8] that when examining amalgamation it is sufficient to prove so-called two-point-amalgamation when trying to show that the amalgamation property holds for a specific class of structures. This will be especially useful in the next chapter when some examples for permutation pattern avoidance classes are studied.

The reason that amalgamation classes are very useful is that they guarantee the existance of a homogeneous limit structure. From this property many other properties can be deduced.

In 2002 Cameron investigated which permutation pattern avoidance classes do have the joint embedding property and amalgamation property [7]. He noted that in the case of permutation pattern avoidance any class that has the amalgamation property does have the joint embedding property as well.

The result Cameron found is as follows. The proof for it can be found in [7], and will be omitted here.

▶ **Theorem 2.31 – Cameron, 2002**
A class of finite permutations is a Fräissé class if and only if it is one of the following:

- Av(∅), that is, the class of all finite permutations,
- Av(21), that is, the class of all identity permutations,
- Av(12), that is, the class of all reversals of identity permutations,
- Av(12, 21), that is, the class that only contains the identity permutation of length 1,
- Av(132, 213), that is, the class of decreasing sequences of increasing sequences,
- Av(231, 312), that is, the class of increasing sequences of decreasing sequences.

This leaves many permutation pattern avoidance classes for which it is still not known whether they have some kind of universal limit akin to a Fraïssé limit.

2.2.3 ω-categorical and Model-complete Theories

For the notions introduced in this section, it will be assumed that one knows about building a first-order language from a given signature L, and in particular what first-order sentences on a L are. A thorough introduction of these notions can be found in [13], but also in almost any other literature on the fundamentals of model theory.

▻ **Definition 2.32 – Theory, model**
Let L be a signature. A set of first-order sentences on L is a *first-order theory* on L.

Let $\mathcal{A}$ be an L-structure. For a sentence ϕ on L, ϕ is said to be true in $\mathcal{A}$, denoted $\mathcal{A} \models \phi$, if ϕ is true for any variable assignment in $\operatorname{dom}(\mathcal{A})$.

Let T be a theory on L. If $\mathcal{A} \models \phi$ holds for all $\phi \in T$, then $\mathcal{A}$ is a *model* of T. $\operatorname{Th}(\mathcal{A})$ denotes the *complete first-order theory* of $\mathcal{A}$, that is, the set of all first-order sentences that hold in $\mathcal{A}$. A theory on L of the form $\operatorname{Th}(\mathcal{A})$ for some L-structure $\mathcal{A}$ is called *complete*. Analogously, for a class of L-structures $\mathscr{A}$, $\operatorname{Th}(\mathscr{A})$ denotes the set of all first-order sentences that hold in all structures in $\mathscr{A}$.

From now on all theories will be assumed to be first-order theories, so the first-order will be dropped from time to time.

Having the notion of a theory and models of a theory it is possible to move on to some more advanced concepts. The following definition is going to be essential for several of the examples in the next chapter.

▻ **Definition 2.33 – ω-categorical**
A complete theory is said to be *ω-categorical* if it has exactly one countable model up to isomorphism. If for a structure $\mathcal{A}$, $\operatorname{Th}(\mathcal{A})$ is ω-categorical, $\mathcal{A}$ is also said to be ω-categorical.

Note that in some literature, instead of ω-categorical, the term $\aleph_0$-categorical is used.

To determine whether a structure is ω-categorical, it is quite useful to have some kind of characterisation. The following is an extract of a theorem associated with Engeler, Ryll-Nardzewski, and Svenonius, who independently of each other proved different versions of it in 1959. The literature sometimes refers to it as the RYLL-NARZEWSKI theorem. A version covering, among others, the characterisations of ω-categorical listed below can be found as Theorem 6.3.1 in [13]. There, a detailed proof is given as well.

▻ Theorem 2.34 – Engeler, Ryll-Nardzewski, Svenonius
Let L be a countable or finite signature and $\mathcal{A}$ a countably infinite L-structure. Then the following are equivalent:

1. $\mathcal{A}$ is ω-categorical.
2. $\mathrm{Aut}(\mathcal{A})$ has finitely many orbits on $\mathrm{dom}\,(\mathcal{A})^n$ for all $n \in \mathbb{N}$, that is to say, $\mathrm{Aut}(\mathcal{A})$ is *oligomorphic*.
3. For each $n \in \mathbb{N}$, every n-type of $\mathrm{Th}\,(\mathcal{A})$ is principal.

Concerning the thesis at hand, the most important characterisation of ω-categorical is the second item. The following two lemmata follow directly from it. A detailed proof for the first can be found in [17], while a proof of the second can be found in [21].

▻ Lemma 2.35
Let $\mathcal{A}$ be a homogeneous structure over a finite relational signature. Then $\mathcal{A}$ is ω-categorical.

▻ Lemma 2.36
Let L be a finite relational signature, and $\mathcal{A}$ an ω-categorical L-structure. Let $L' \subseteq L$. Then the L'-reduct of $\mathcal{A}$ is ω-categorical.

Another useful notion is that of model-completeness and model companions as introduced below.

▻ Definition 2.37 – Model-complete, model companion
Let L be a signature. A first-order theory on L is said to be *model-complete* if every embedding between its models is elementary, that is, preserves all first-order formulas.

Let T be a theory on L. Then a theory U on L is a *model companion* of T if U is model-complete, and every model of T has an extension which is a model of U, and vice versa. If T does have a model companion it is said to be *companionable*.

It was shown by ROBINSON and BARWISE [18] that if a model companion exists it is unique up to logical equivalence. The following theorem taken from [4] is an extract of Theorem 3.6.7 there and presents a way of determining whether a theory is model-complete. The proof will be omitted here. But before it is actually stated, another definition is necessary.

▻ Definition 2.38 – Existentially definable
Let $\mathcal{A}$ be an L-structure, and R a relation symbol of arity n that does not occur in L. Let $L' = L \cup R$, and $\mathcal{A}'$ be an expansion of $\mathcal{A}$ to L'. Then R is said to be *existentially definable* in $\mathcal{A}$ if there exists a first-order formula $\phi(x_1, \ldots, x_n)$ on L with n free variables $x_1, \ldots, x_n$ that does not contain the universal quantifier $\forall$ and only non-nested negation such that for any $a_1, \ldots, a_n \in \mathrm{dom}\,(\mathcal{A})$, $(a_1, \ldots, a_n) \in R$ if and only if $\phi(a_1, \ldots, a_n)$ is true.

▶ **Theorem 2.39 – Bodirsky**
Let $\mathcal{A}$ be an ω-categorical L-structure. Then the following statements are equivalent:

1. $\mathrm{Th}(\mathcal{A})$ is model-complete.
2. There exists a homogeneous expansion of $\mathcal{A}$ by countably many relations $R_1, R_2, \ldots$ such that for all i, R_i and its complement are existentially definable in $\mathcal{A}$.
3. Every self-embedding of $\mathcal{A}$ is locally generated by the automorphisms of $\mathcal{A}$.

In particular, the complete first-order theory of a homogeneous, ω-categorical structure is model-complete.

Another interesting result is due to Saracino. Together with a detailed proof, the following theorem can be found in [20].

▶ **Theorem 2.40 – Saracino**
Let T be an ω-categorical theory that has no finite models. Then T has a model companion U, and U is ω-categorical as well.

In particular, this means that if an ω-categorical theory has a countably infinite model, then it has an ω-categorical model companion. This will be very useful for most examples in the following chapter.

3 Patterns of Length 3

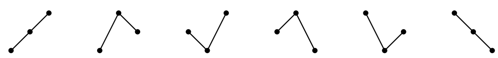
In this chapter the results found for permutation pattern avoidance classes that avoid sets of patterns of length 3 are presented. As already mentioned in the introduction the focus will be on the model theoretic properties of each class. For a detailed survey concerning the combinatorial properties of these avoidance classes in particular see [22].

Before going into depth, note that patterns will not always be given explicitly, but may instead be stated using pictograms. In these cases the correlation between patterns and graphical interpretations as defined in the previous chapter can be used to switch between both.

3.1 Symmetries

The first thing one should note is that symmetric permutation pattern avoidance classes do not only have the same combinatorial properties, but also share their model theoretic properties. This is due to the fact that all symmetries can be achieved by switching around the naming of $<_1$ and $<_2$ or reversing the definition of one or both of the strict total orders. This does obviously not change the model theoretic properties, though. Hence, the first step to take is to identify non-symmetric permutation pattern avoidance classes.

To find all non-symmetric sets of avoidable patterns it is necessary to find all patterns of length 3 first. Obviously, the following six patterns are the only ones, as they correspond to all possible orderings of the numbers 1, 2 and 3:



123 132 213 231 312 321

In Section 2.1.2 it was shown that two avoidance classes avoiding the sets of patterns $\mathcal{F}$ and $\mathcal{F}'$, respectively, are symmetric if there is a transformation $\tau \in D_4$ such that $\mathcal{F}' = \tau(\mathcal{F})$. Table 3.1 illustrates the symmetries of the stated patterns of length 3. By that one can easily identify the symmetric avoidance classes. This reduces the need for individual analysis to the 21 classes given in Table 3.2.

The class avoiding no patterns of length 3 will not be discussed in an extra section. It obviously contains all permutations, and is therefore known to be an amalgamation class [7].

The following sections are ordered by the number of avoided patterns. If not stated otherwise, the surveyed classes will be the ones avoiding the sets identified in Table 3.2.

Table 3.1. Symmetries of each pattern of length 3 with respect to the symmetry group of a square.

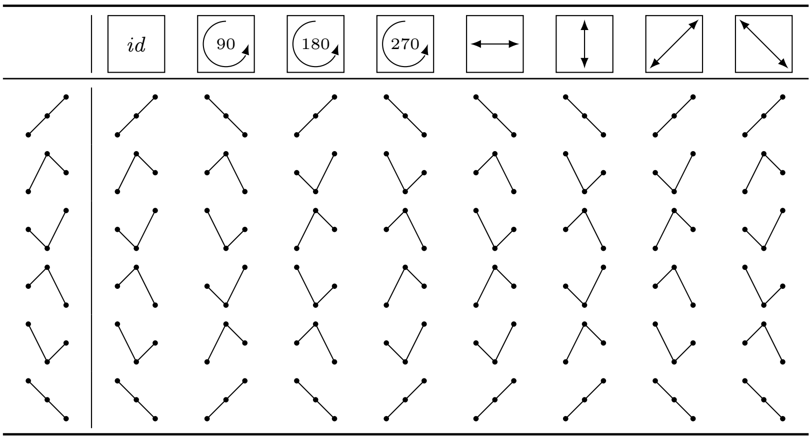
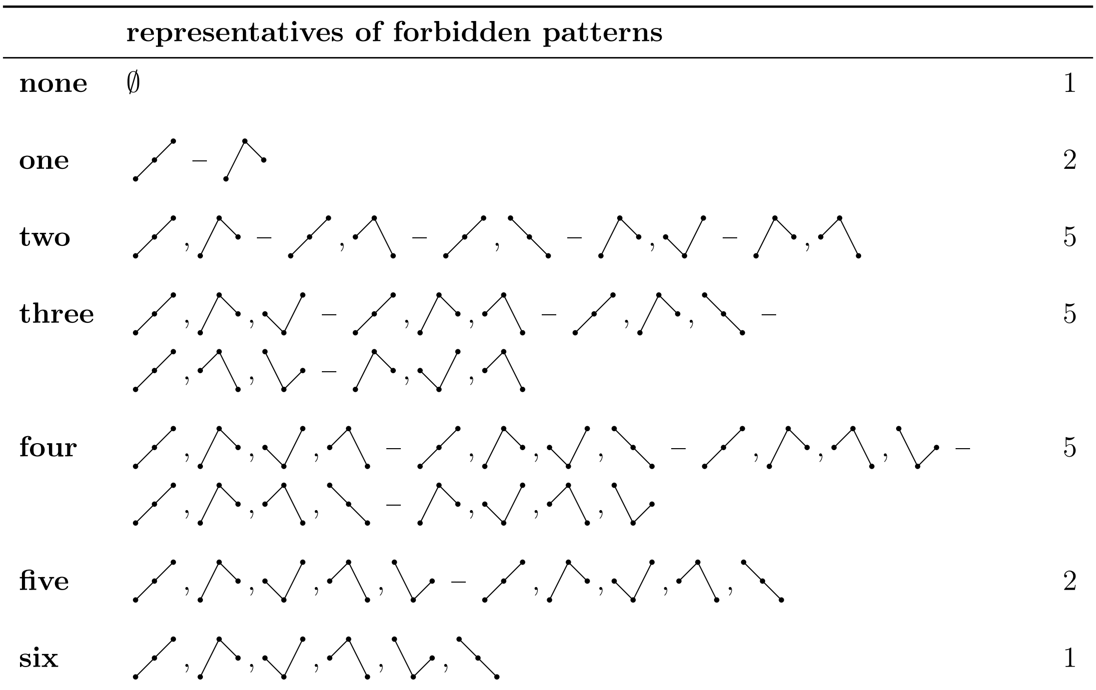


Table 3.2. A representative for each set of symmetric sets of avoided patterns of length 3 sorted by number of avoided patterns.



3.2 Avoiding a single pattern

Although there are six permutations of length 3 they belong to two non-symmetric classes. These are as follows.

- $\{123\}, \{321\}$
- $\{132\}, \{213\}, \{231\}, \{312\}$

For the following parts of this section the avoidance sets $\{123\}$ and $\{231\}$ were chosen as representatives and looked into in more detail.

It was shown by MacMahon [16] that the class Av(123) is enumerated by the Catalan numbers. Later Knuth [15] noticed that the permutations avoiding the pattern 231 are precisely the stack-sortable permutations. He proved that this class is enumerated by the Catalan numbers as well. That is, since Av(132) and Av(231) are symmetric,

$$|\mathrm{Av}_n(123)| = |\mathrm{Av}_n(132)| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

for all $n \in \mathbb{N}$. Therefore, all permutation pattern avoidance classes avoiding exactly one pattern of length 3 are Wilf-equivalent.

To date, several bijections between both classes have been found. One of the better known bijections, and also one of the first, is due to Simion and Schmidt [22]. A comprehensive survey comparing known bijections has been published in 2008 by Claesson and Kitaev [9], the interest of which lies in examining the properties that are preserved by the various bijections.

3.2.1 Class Av(123)

For this class it is relatively easy to see that the joint embedding property holds by taking a look at whether the forbidden permutation is $\oplus$- or $\ominus$-indecomposable.

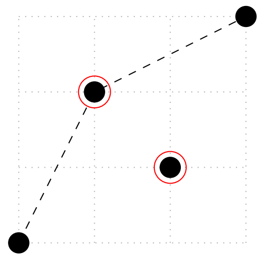
▻ Lemma 3.1
The permutation pattern avoidance class Av(123) has the joint embedding property.

Proof. As the permutation 123 is $\ominus$-indecomposable, this follows directly from Corollary 2.18 in conjunction with Corollary 2.16 and the fact that the notion of atomic is equivalent to a class having the joint embedding property. □

It is known from Theorem 2.31 due to Cameron [7], that Av(123) does not have the amalgamation property. An example for the amalgamation not working in this class can be seen in Figure 3.1.

So far, it has unfortunately not been possible to prove or disprove the following, but it seems very likely that it holds.

(a) Permutation 132 and 213 with the common pattern 21.



(b) Only possible amalgam of 132 and 213 preserving the common pattern 21. Unfortunately, the amalgam, 1324, contains the pattern 123.

Figure 3.1. Counterexample showing that the permutation class Av(123) does not have the amalgamation property.

▻ Conjecture 3.2
The complete first-order theory Th (Av(123)) of Av(123) has an ω-categorical model companion.

3.2.2 Class Av(231)

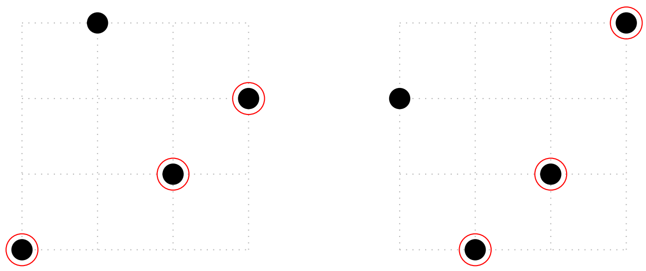
As for the previous class it is relatively straightforward to spot that the permutation pattern avoidance class Av(231) has the joint embedding property.

▻ Lemma 3.3
Av(231) has the joint embedding property.

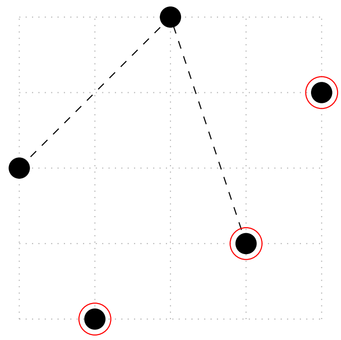
Proof. Note that, since a permutation pattern avoidance class has the joint embedding property if and only if it is atomic, it is sufficient to prove that Av(231) is atomic. Since 231 is $\oplus$-indecomposable, Av(231) has the joint embedding property according to Lemma 2.17 in conjunction with Lemma 2.15. □

According to Cameron [7] this class does not have the amalgamation property. An example that falsifies the amalgamation property in the case of Av(231) can be seen in Figure 3.2.

As mentioned earlier, this class can be enumerated by the Catalan numbers. Over time, several bijections between this class and other structures enumerated by the Catanlan numbers have been given, one of which relates Av(231) to binary trees [14]. But although this seemed very



(a) Two permutations of length 4 containing the pattern 123, namely, 1423 and 3124.



(b) Only possible amalgam of 1423 and 3124 preserving the pattern 123. The amalgam, 31524, contains the pattern 231, though.

Figure 3.2. Counterexample showing that the permutation class Av(231) does not have the amalgamation property. This is an adaptation of an example found in [7].

promising for obtaining an amalgamation class of which Av(231) is the $\{<_1, <_2\}$-reduct, this has so far not been achieved. Therefore, the following remains a conjecture in this case as well.

▻ **Conjecture 3.4**
The complete first-order theory $\mathrm{Th}\,(\mathrm{Av}(231))$ of Av(231) has an ω-categorical model companion.

3.3 Avoiding two patterns

According to Table 3.1 the fifteen possible choices can be allocated to five sets of symmetric sets of two patterns of length 3. These are as follows.

- $\{123, 132\}$, $\{123, 213\}$, $\{231, 321\}$, $\{312, 321\}$
- $\{123, 231\}$, $\{123, 312\}$, $\{132, 321\}$, $\{213, 321\}$
- $\{123, 321\}$

- $\{132, 213\}$, $\{231, 312\}$
- $\{132, 231\}$, $\{132, 312\}$, $\{213, 231\}$, $\{213, 312\}$

Note that combinatorially there are only three Wilf-equivalence classes. It is known [22] that

$$|\mathrm{Av}_n(123, 132)| = |\mathrm{Av}_n(132, 213)| = |\mathrm{Av}_n(132, 231)| = 2^{n-1}$$
$$|\mathrm{Av}_n(123, 231)| = 1 + \binom{n}{2}$$

for all $n \in \mathbb{N}$, and

$$|\mathrm{Av}_n(123, 321)| = \begin{cases} n, & \text{if } n = 1, 2 \\ 4, & \text{if } n = 3, 4 \\ 0, & \text{if } n \geq 5 \end{cases} .$$

The following sections now discuss the model theoretic properties of each of the five symmetry classes. The chosen representatives are $\{123, 132\}$, $\{123, 231\}$, $\{123, 321\}$, $\{132, 213\}$ and $\{132, 312\}$.

3.3.1 Class Av(123, 132)

Again it is relatively easy to prove that the class has the joint embedding property.

▻ Lemma 3.5
The class $\mathrm{Av}(123, 132)$ has the joint embedding property.

Proof. By Corollary 2.18 it is known that $\mathrm{Av}(123, 132)$ is $\ominus$-sum-complete since 123 and 132 are $\ominus$-indecomposable. Therefore, by Corollary 2.16, this class does have the joint embedding property. □

Before going further into detail concerning the model theoretic properties it is reasonable to take a closer look at the general structure of the permutations in this class. This reveals that the following lemma holds.

▻ Lemma 3.6
For each permutation in $\mathrm{Av}(123, 132)$ there exists a unique representation as a finite skew sum of permutations of the form $id_n^r \oplus 1$, $n \in \mathbb{N}_0$.

Furthermore, each skew sum of finitely many permutations of the form $id_n^r \oplus 1$, $n \in \mathbb{N}_0$, is isomorphic to a permutation avoiding the patterns 123 and 132.

Proof. Let $\pi \in \mathrm{Av}(123, 132)$. Then let $\pi_1 \ominus \pi_2 \ominus \ldots \ominus \pi_m$ be the $\ominus$-decomposition of π into $\ominus$-indecomposable permutations. Suppose that for some $i \in [m]$, π_i is not of the stated form. Obviously, $|\pi_i| > 2$ since all permutations of length 1 or 2 are either of the desired form or $\ominus$-decomposable.

Without loss of generality it can be assumed that π_i is of the form $p_1p_2 \dots p_k$ where $k = |\pi_i|$. Now let s be the index of the smallest entry of π_i, and t be the index of the largest. That is, $p_s = 1$ and $p_t = k$. Since π_i is $\ominus$-indecomposable, it holds that $s < k$ and $t > 1$. There are now two cases to consider. Either $s < t$, or $t < s$.

Case 1: $s < t$. In this case s and t have to be consecutive since otherwise π_i would contain an increasing subsequence of length 3 as for $t-1 > s$ it is $1 = p_s < p_{s+1} < p_t = k$. Furthermore, $t = k$ since in any other case π_i would contain a subsequence isomorphic to 132. Thus, $s = k-1$ and $t = k$.

For p_i to not be of the form id_n^r for some $n \in \mathbb{N}_0$, the remaining $k-2$ entries $p_1 \dots p_{k-2}$ cannot be strictly decreasing. Therefore, there exists $j \in [k-3]$ with $p_j < p_{j+1}$. Then $p_j < p_{j+1} < p_k$ holds. Since $j < k$ this implies that π_i contains an increasing subsequence of length 3. This is a contradiction with the premise that p_i avoids 123.

Case 2: $t < s$. Since $s < k$ and $1 < t$ this implies that neither p_1 nor p_k are 1 or k, and $1 < t < k-1$. Let u be the smallest entry in the sequence $p_1 \dots p_{t-1}$. This is well-defined as $t > 1$, and it is $1 < u < k$. Analogously, let v be the largest entry in the sequence $p_{t+1} \dots p_k$. Since $t < k-1$ this is also well-defined, and it is $1 < v < k$.

Obviously, ukv is a subsequence of π_i. Due to π_i avoiding the pattern 132, and k being the biggest entry in π_i and its subsequence ukv in particular, ukv is isomorphic to 231. But then, since u is the smallest entry in $p_1 \dots p_{t-1}$ and v the largest in $p_{t+1} \dots p_k$, π_i is $\ominus$-decomposable, namely into the permutation isomorphic to $p_1 \dots p_t$ and the one isomorphic to $p_{t+1} \dots p_k$. This contradicts the assumption that π_i is $\ominus$-indecomposable.

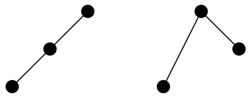
Therefore, neither of the two cases can hold, and so the assumption is falsified. Thus, each component in the $\ominus$-decomposition of π is indeed of the form $id_n^r \oplus 1$, $n \in \mathbb{N}_0$.

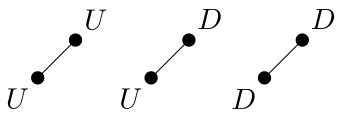
For the second part of the lemma, let the permutation π be the skew sum of finitely many permutations of the desired form. Suppose that $\pi \notin \mathrm{Av}(123, 132)$. Hence, π contains the pattern 123 or the pattern 132.

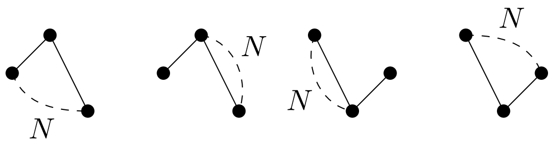
If $123 \preceq \pi$ then one of the components upon $\ominus$-decomposition has to contain the pattern 123 as this is $\ominus$-indecomposable. This contradicts the form of these components, since any permutation of the form $id_n^r \oplus 1$ does not contain 123. On the other hand, if $132 \preceq \pi$, then once again one of the components has to contain this pattern. In the same way as for 123 this cannot be the case since any permutation of the form $id_n^r \oplus 1$ avoids the pattern 132. Therefore, the stated assumption cannot be valid, so instead $\pi \in \mathrm{Av}(123, 132)$ holds. □

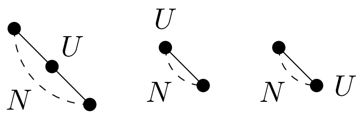
It is now necessary to look at a slightly different structure, or more specifically at an expansion of the permutation pattern avoidance class at hand. That the class defined in the following theorem indeed expands $\mathrm{Av}(123, 132)$ will be shown later on.

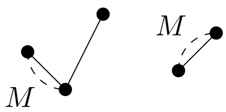
▶ Theorem 3.7
Let $\mathcal{X}$ be an L-structure with domain X and relational signature $L = \{<_1, <_2, D, U, M, N\}$, where D, U are unary, and $<_1, <_2, M, N$ binary relations, that avoid the patterns

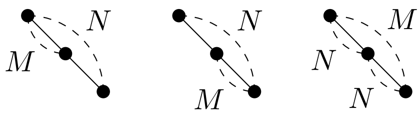
(R1) 

(R2) 

(R3) 

(R4) 

(R5) 

(R6) 

and for which

- $<_1, <_2$ are strict total orders,
- $U \cap D = \emptyset$, $U \cup D = X$,
- $M \cap N = \emptyset$, $M \cup N = X \times X$, M, N symmetric.

Then the class $\mathscr{X}$ of all finite L-structures with the properties given above is an amalgamation class.

Proof. Note that, as it seems obvious, it will not always be mentioned explicitly that if for any a, b it holds that $(a, b) \in M$, or N, respectively, then due to the symmetric nature of the relation, $(b, a) \in M$, or N, respectively, will be assumed to hold as well.

To prove that $\mathscr{X}$ is an amalgamation class, it is necessary and sufficient by Theorem 2.30 to show that $\mathscr{X}$ is closed under isomorphism, and has the hereditary property, joint embedding property, and amalgamation property.

Since the structures in $\mathscr{X}$ are defined by avoiding certain substructures, the class is closed under isomorphism, and does have the hereditary property.

It is not as easy to see that the joint embedding property holds. Let $\mathcal{X}, \mathcal{Y} \in \mathscr{X}$ with $X = \operatorname{dom}(\mathcal{X})$, $Y = \operatorname{dom}(\mathcal{Y})$. Without loss of generality it can be assumed that X and Y are disjoint. If this would not be the case and they share an element x, mark this element with 1 in X and with 2 in Y. This does not change anything about the properties of the elements of each respective set.

Let $\mathcal{Z}$ be the L-structure defined on the domain $Z = X \cup Y$ with interpretations of the relations such that for $u, v \in Z$

$$\begin{aligned}
<_1^{\mathcal{Z}} &:= <_1^{\mathcal{X}} \cup <_1^{\mathcal{Y}} \cup (X \times Y), \\
<_2^{\mathcal{Z}} &:= <_2^{\mathcal{X}} \cup <_2^{\mathcal{Y}} \cup (Y \times X), \\
D^{\mathcal{Z}} &:= D^{\mathcal{X}} \cup D^{\mathcal{Y}}, \\
U^{\mathcal{Z}} &:= U^{\mathcal{X}} \cup U^{\mathcal{Y}}, \\
M^{\mathcal{Z}} &:= M^{\mathcal{X}} \cup M^{\mathcal{Y}} \cup (X \times Y) \cup (Y \times X), \\
N^{\mathcal{Z}} &:= N^{\mathcal{X}} \cup N^{\mathcal{Y}}.
\end{aligned}$$

Obviously, $\mathcal{Z}$ embeds both, $\mathcal{X}$ and $\mathcal{Y}$. To prove that $\mathcal{Z} \in \mathscr{X}$ holds as well, it needs to be verified that $\mathcal{Z}$ satisfies (R1) to (R6), and the interpretations of the relations have the stated properties.

By the definitions stated above, $<_1^{\mathcal{Z}}$ and $<_2^{\mathcal{Z}}$ are strict total orders, and M and N are symmetric. Furthermore, using that X and Y are disjoint yields that

$$\begin{aligned}
U^{\mathcal{Z}} \cap D^{\mathcal{Z}} &= (U^{\mathcal{X}} \cup U^{\mathcal{Y}}) \cap (D^{\mathcal{X}} \cup D^{\mathcal{Y}}) \\
&= (U^{\mathcal{X}} \cap D^{\mathcal{X}}) \cup (U^{\mathcal{X}} \cap D^{\mathcal{Y}}) \cup (U^{\mathcal{Y}} \cap D^{\mathcal{X}}) \cup (U^{\mathcal{Y}} \cap D^{\mathcal{Y}}) \\
&= \emptyset \cup \emptyset \cup \emptyset \cup \emptyset = \emptyset, \\
U^{\mathcal{Z}} \cup D^{\mathcal{Z}} &= (U^{\mathcal{X}} \cup U^{\mathcal{Y}}) \cup (D^{\mathcal{X}} \cup D^{\mathcal{Y}}) \\
&= (U^{\mathcal{X}} \cup D^{\mathcal{X}}) \cup (U^{\mathcal{Y}} \cup D^{\mathcal{Y}}) = X \cup Y = Z, \\
M^{\mathcal{Z}} \cap N^{\mathcal{Z}} &= (M^{\mathcal{X}} \cup M^{\mathcal{Y}} \cup (X \times Y) \cup (Y \times X)) \cap (N^{\mathcal{X}} \cup N^{\mathcal{Y}}) \\
&= (M^{\mathcal{X}} \cap N^{\mathcal{X}}) \cup (M^{\mathcal{X}} \cap N^{\mathcal{Y}}) \cup (M^{\mathcal{Y}} \cap N^{\mathcal{X}}) \cup (M^{\mathcal{Y}} \cap N^{\mathcal{Y}}) \\
&\quad \cup ((X \times Y) \cap N^{\mathcal{X}}) \cup ((X \times Y) \cap N^{\mathcal{Y}}) \\
&\quad \cup ((Y \times X) \cap N^{\mathcal{X}}) \cup ((Y \times X) \cap N^{\mathcal{Y}}) \\
&= \emptyset \cup \emptyset \cup \emptyset \cup \emptyset \cup \emptyset \cup \emptyset \cup \emptyset \cup \emptyset = \emptyset, \\
M^{\mathcal{Z}} \cup N^{\mathcal{Z}} &= (M^{\mathcal{X}} \cup M^{\mathcal{Y}} \cup (X \times Y) \cup (Y \times X)) \cup (N^{\mathcal{X}} \cup N^{\mathcal{Y}}) \\
&= (M^{\mathcal{X}} \cup N^{\mathcal{X}}) \cup (M^{\mathcal{Y}} \cup N^{\mathcal{Y}}) \cup (X \times Y) \cup (Y \times X) \\
&= (X \times X) \cup (X \times Y) \cup (Y \times X) \cup (Y \times Y) \\
&= (X \times (X \cup Y)) \cup (Y \times (X \cup Y)) = ((X \cup Y) \times (X \cup Y)) = Z \times Z.
\end{aligned}$$

Due to the definitions of the relations, a violation of one of the restraints can only occur if some of the elements in question were originally in X, and some in Y. Note that for any two elements $x, y \in Z$ with $x \in X$ and $y \in Y$ it is $x <_1 y$ and $y <_2 x$, and $(x, y) \in M$. Therefore, if any restraint is not violated in $\mathcal{X}$ or $\mathcal{Y}$, then it is not violated in $\mathcal{Z}$ either. Since $\mathcal{X}, \mathcal{Y} \in \mathscr{X}$, it follows that $\mathcal{Z} \in \mathscr{X}$ holds. Hence, the class $\mathscr{X}$ has the joint embedding property.

Now the only remaining property is the amalgamation property. Luckily, to prove the amalgamation property it is sufficient to consider so-called two-point-amalgamation [8] and prove that $\mathscr{X}$ has this property.

Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y} \in \mathscr{X}$ such that $\mathcal{Y}$ is a substructure of $\mathcal{X}_1$ as well as $\mathcal{X}_2$, and

$$X_1 = \operatorname{dom}(\mathcal{X}_1), \qquad X_2 = \operatorname{dom}(\mathcal{X}_2), \qquad Y = \operatorname{dom}(\mathcal{Y}),$$
$$X_1 = Y \cup \{x_1\}, \qquad X_2 = Y \cup \{x_2\}.$$

Then for $<_1$, and $<_2$, respectively, there are 3 possibilities as to how x_1 and x_2 are related to each other. For $<_1$ they are:

(A) $\exists\, x : x_1 <_1 x <_1 x_2$,

(B) $\exists\, x : x_2 <_1 x <_1 x_1$,

(C) $\forall\, x : (x <_1 x_1 \wedge x <_1 x_2) \vee (x_1 <_1 x \wedge x_2 <_1 x)$.

Analogously, for $<_2$ they are:

(a) $\exists\, y : x_1 <_2 y <_2 x_2$,

(b) $\exists\, y : x_2 <_2 y <_2 x_1$,

(c) $\forall\, y : (y <_2 x_1 \wedge y <_2 x_2) \vee (x_1 <_2 y \wedge x_2 <_2 y)$.

So combining these yields the nine possible cases Aa, Ab, Ac, Ba, Bb, Bc, Ca, Cb, and Cc.

Obviously, the cases Aa and Bb are symmetric as one can be obtained from the other by exchanging the roles of $\mathcal{X}_1$ and $\mathcal{X}_2$. Analogously, cases Ab and Ba, Ac and Bc, and cases Ca and Cb are symmetric. Next, note that the set of restraints (R1) to (R6) is symmetric to itself under application of $\tau \in D_4$ where τ is the reflection about the increasing diagonal. This way, case Ba can be reduced to case Ab. In the same way, Ca can be reduced to Ac, and Cb to Bc. Hence, it suffices to look at the four cases

1. Aa, which also covers Bb,
2. Ab, which also covers Ba,
3. Ac, which also covers Bc, Ca, and Cb,
4. Cc.

***Case 1** (Aa):*

$$\exists\, x : x_1 <_1 x <_1 x_2 \tag{7}$$
$$\exists\, y : x_1 <_2 y <_2 x_2 \tag{8}$$

This implies directly that $x_1 <_1 x_2$ as well as $x_1 <_2 x_2$. For this to work none of the following may hold:

$$x_1 \in U, \tag{9}$$
$$x_2 \in D, \tag{10}$$
$$\exists\, r \in Y : x_1 <_1 r \wedge r <_1 x_2 \wedge x_2 <_2 r \tag{11}$$
$$\exists\, s \in Y : x_2 <_1 s \wedge x_1 <_2 s \wedge x_2 <_2 s \tag{12}$$

Assumption 1: Suppose that (9) holds, that is $x_1 \in U$. Then due to restriction (R2) it has to be $x <_2 x_1$ and $y <_1 x_1$ in $\mathcal{X}_1$. By the premises of this case, it is therefore $x <_2 x_2$ and $y <_1 x_2$. Thus, by (R5) it is $(y, x) \in N$ in $\mathcal{X}_2$. On the other hand, as x and y belong to the shared structure $\mathcal{Y}$, $(y, x) \in N$ has to hold in $\mathcal{X}_1$ as well. But this contradicts the (R4) in $\mathcal{X}_1$.

Assumption 2: Suppose that (10) holds, that is $x_2 \in D$. From the premises of this case and (R2) it is known that $x_2 <_1 y$ and $x_2 <_2 x$. This also implies $x_1 <_1 y$ and $x_1 <_2 x$. But then $\mathcal{X}_1$ would contain the pattern 132 since this is isomorphic to the sequence (x_1, x, y) in this case, which contradicts the restraint (R1).

Assumption 3: Suppose that (11) holds, that is there exists $r \in Y$ such that $x_1 <_1 r$, $r <_1 x_2$, and $x_2 <_2 r$. Then it would be $r \in U$ due to (R2) in $\mathcal{X}_1$, and $y <_1 x_1$ as well as $y \in D$ because of (R1) and (R2) in $\mathcal{X}_1$. The latter in combination with (8) implies $y <_1 x_2$. Thus, a contradiction occurs in $\mathcal{X}_2$ since (R1) is violated.

Assumption 4: Suppose that (12) holds. This yields a contradiction analogously to Assumption 3, which can be obtained from this one by exchanging the roles of $<_1$ and $<_2$.

Since all above assumptions can be falsified, it is possible to amalgamate $\mathcal{X}_1$ and $\mathcal{X}_2$ by extending $\mathcal{X}_1$ by x_2 and adding x_2 to U, (x_1, x_2) to $<_1$, (x_1, x_2) to $<_2$, and $(x_1, x_2), (x_2, x_1)$ to N. This amalgamation satisfies all restraints.

***Case 2** (Ab):*

$$\exists\, x : x_1 <_1 x <_1 x_2 \tag{13}$$
$$\exists\, y : x_2 <_2 y <_2 x_1 \tag{14}$$

From these premises it follows directly that $x_1 <_1 x_2$ and $x_2 <_2 x_1$. Only judging from this any assignment of x_1, x_2 to U and D is feasible. So there are now four cases to consider.

Case 2.1: $x_1, x_2 \in U$. In this case it is known from the restraints that $x <_2 x_1$, $y <_1 x_2$. Then the only way a problem could occur is if

$$\exists\, z : z <_1 x_1 \wedge z <_2 x_2 \tag{15}$$

Now suppose that (15) holds. Then in $\mathcal{X}_1$ to avoid (R1) it is $x <_2 z$. Furthermore, $(z, x) \in M$ due to (R3). But this implies that in $\mathcal{X}_2$ the restraint (R5) is violated.

Therefore, the assumption cannot hold, and $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate in the canonical way with $(x_1, x_2), (x_2, x_1) \in M$.

Case 2.2: $x_1, x_2 \in D$. As this case is slightly more complicated than the previous one it will be split up into three sub-cases:

$$\exists\, r : x_1 <_1 r \wedge r <_1 x_2 \wedge x_1 <_2 r,$$
$$\exists\, s : x_2 <_2 s \wedge s <_2 x_1 \wedge x_2 <_1 s,$$
$$\forall\, r : ((x_1 <_1 r \wedge r <_1 x_2) \to r <_2 x_1) \wedge ((x_2 <_2 r \wedge r <_2 x_1) \to r <_1 x_2).$$

Obviously, the first two of these may overlap. They were chosen this way for symmetric reasons. It is also easy to see that no other cases may occur.

Case 2.2.1: $\exists\, r : x_1 <_1 r \wedge r <_1 x_2 \wedge x_1 <_2 r$. Considering $\mathcal{X}_1$ this means that due to (R2) $r \in U$ and because of (R5) $(x_1, r) \in N$. Therefore, to satisfy restraint (R3) for the amalgam it has to be $(x_1, x_2) \in M$. There are three ways this may not work out. They are covered by the following assumptions, and their respective falsification.

Assumption 1: Suppose that there exists $z \in Y$ such that $z <_1 x_1$, $x_1 <_2 z$, and $(z, x_2) \in N$. Then because of (13) and (14), obviously $z <_1 x_2$, and $x_2 <_2 z$. So now two cases are possible. If $z <_2 r$ then $\mathcal{X}_2$ would not be an admissible structure as it would violate restraint (R3). On the other hand, if $r <_2 z$ then (R4) is not satisfied in $\mathcal{X}_2$. Thus, the assumption cannot hold.

Assumption 2: Suppose that there exists $z \in Y$ such that $z <_2 x_2$, $x_2 <_1 z$, and $(x_1, z) \in N$. Then restraint (R3) is violated in $\mathcal{X}_1$. So this cannot be the case.

Assumption 3: Suppose that there exists $z \in Y$ such that $x_2 <_1 z$, and $x_1 <_z$. In this case, (R1) would not be satisfied in $\mathcal{X}_1$. Therefore, this assumption is falsified.

Since none of the assumptions provides a problem, $\mathcal{X}_1$ and $\mathcal{X}_2$ can in this case be amalgamated in the canonical way adding $(x_1, x_2), (x_2, x_1) \in M$.

Case 2.2.2: $\exists\, s : x_2 <_2 s \wedge s <_2 x_1 \wedge x_2 <_1 s$. In this case the same as in Case 2.2.1 applies if one exchanges the roles of $<_1$ and $<_2$. Therefore, $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate analogously when adding in $(x_1, x_2), (x_2, x_1) \in M$.

Case 2.2.3: $\forall\, r : ((x_1 <_1 r \wedge r <_1 x_2) \to r <_2 x_1) \wedge ((x_2 <_2 r \wedge r <_2 x_1) \to r <_1 x_2)$. Then in particular $x <_2 x_1$ and $y <_1 x_2$ because of (13) and (14). Since $x_1, x_2 \in D$ this means that there exists $t \in Y$ such that $x_1 <_1 t <_1 x_2$ and $x_2 <_2 t <_2 x_1$.

Now, if $t \in U$ then to satisfy (R4) in $\mathcal{X}_1$ it would be $(x_1, x) \in M$. Analogously, $(x, x_2) \in M$. Then canonically amalgamating $\mathcal{X}_1$ and $\mathcal{X}_2$ would work when adding $(x_1, x_2), (x_2, x_1) \in M$.

On the other hand, if for all such t it would be $t \in D$, then due to the restraints (R1) to (R6) $\mathcal{X}_1$ and $\mathcal{X}_2$ still amalgamate in the canonical way. Only that if $(x_1, t) \in M$ or $(t, x_2) \in M$ then $(x_1, x_2), (x_2, x_1) \in M$ would need to be added, and if both $(x_1, t) \in N$ and $(t, x_2) \in N$ then $(x_1, x_2), (x_2, x_1) \in N$ would need to be added.

Case 2.3: $x_1 \in D, x_2 \in U$. Due to (R1) this implies $x_1 <_1 y <_1 x_2$. It also means that the only hope for successful amalgamation is by adding $(x_1, x_2), (x_2, x_1) \in M$. Now, if one assumes that there exists $z \in Y$ such that $x_2 <_1 z$, $z <_2 x_2$, and $(x_1, z) \in N$, then this would yield a contradiction. But if such z exists this implies that due to (R6) being satisfied in $\mathcal{X}_1$

it would have to be $(x, z) \in N$ as well. This contradicts (R4) in $\mathcal{X}_2$, though. Therefore, the assumption cannot hold, and $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate accordingly.

Case 2.4: $x_1 \in U, x_2 \in D$. This case is analogous to Case 2.3. Therefore, $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate canonically with $(x_1, x_2), (x_2, x_1) \in M$.

As $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate in all four sub-cases, the amalgamation property is validated for Case 2.

Case 3 *(Ac):*

$$\exists x : x_1 <_1 x <_1 x_2 \tag{16}$$

$$\forall y : (y <_2 x_1 \wedge y <_2 x_2) \vee (x_1 <_2 y \wedge x_2 <_2 y) \tag{17}$$

From (16) it follows that $x_1 <_1 x_2$, and from (17) that either $x_1 <_2 x_2$ or $x_2 <_1 x_1$. It is now possible to consider the following two cases.

Case 3.1: $\exists r \in Y : x_1 <_1 r <_1 x_2 \wedge x_1 <_2 r$. Then from (17) it is known that $x_2 <_2 r$ as well. Then for the amalgam to satisfy (R1) it has to be $x_2 <_2 x_1$. Furthermore, to also satisfy (R3) it has to be $(x_1, x_2), (x_2, x_1) \in M$. Defining this as the amalgam works as any contradiction would directly imply that a restraint violation already occured in $\mathcal{X}_1$ or $\mathcal{X}_2$.

Case 3.2: $\forall r \in Y : (x_1 <_1 r <_1 x_2) \rightarrow (r <_2 x_1)$. In particular, this means that $x <_2 x_1$. Hence, $x <_2 x_2$ by (17), and therefore $x_2 \in U$ because of (R2) in $\mathcal{X}_2$.

Suppose now that $(x_1, x) \in N$. Obviously, this can only hold if $x_1 \in D$ because of (R4). Then for the amalgam to satisfy restraint (R3) it would have to be $x_1 <_2 x_2$. Thus, $(x_1, x_2), (x_2, x_1) \in N$ due to (R5).

Assume that there exists $z \in Y$ such that $z <_1$, $x_2 <_2 z$, and $(x_1, z) \in N$. Then in $\mathcal{X}_1$, $(x, z) \in N$ because of (R6). But this contradicts (R3) in $\mathcal{X}_2$.

Any other assumption that would lead to the amalgam obtained in this way to not be in $\mathscr{X}$ would directly lead to a contradiction in $\mathcal{X}_1$ or $\mathcal{X}_2$ as well. Therefore, $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate in this case.

On the other hand, if for all r with $x_1 <_1 r <_1 x_2$ it is $(x_1, r) \in M$, then $x_2 <_2 x_1$ has to hold to satisfy (R5), and $(x_1, x_2), (x_2, x_1) \in M$ to satisfy (R4). The so obtained structure is admissible, since any contradiction within it would imply that there already was one in either $\mathcal{X}_1$ or $\mathcal{X}_2$.

Therefore, $\mathcal{X}_1$ and $\mathcal{X}_2$ have the amalgamation property if they satisfy the premises of Case 3.

Case 4 *(Cc):*

$$\forall x : (x <_1 x_1 \wedge x <_1 x_2) \vee (x_1 <_1 x \wedge x_2 <_1 x) \tag{18}$$

$$\forall y : (y <_2 x_1 \wedge y <_2 x_2) \vee (x_1 <_2 y \wedge x_2 <_2 y) \tag{19}$$

In this case, the relative position of x_1 and x_2 in the amalgam is not restricted by the structures $\mathcal{X}_1$ and $\mathcal{X}_2$. The trivial case is that both x_1 and x_2 are in U or D, respectively, and that for any $u \in Y$ it is $(u, x_1) \in M$ if and only if $(u, x_2) \in M$. In this case both structures are isomorphic and a valid amalgamation is to take $\mathcal{X}_1$ as the amalgam and identify x_2 with x_1. Now suppose that this is not the case.

If both structures only differ in that $x_1 \in U$, and $x_2 \in D$, or vice versa, then because of restraint (R2) together with (18) and (19) it would hold for all $v \in Y$ that either $v <_1 x_2 \wedge x_1 <_2 v$ or $x_1 <_1 v \wedge v <_2 x_1$, and the same for x_2. From this it follows directly that to satisfy restraint (R2) in $\mathcal{X}_1$ as well as $\mathcal{X}_2$ for all $v \in Y$ it is $(v, x_1) \in M$ and $(v, x_2) \in M$. Therefore, amalgamating both structures by adding $x_1 <_1 x_2$, $x_2 <_2 x_1$, and $(x_1, x_2), (x_2, x_1) \in M$ works.

The case now left is that there exists $r \in Y$ such that (x_1, r) and (x_2, r) are not both in the respective interpretations of M or N. Without loss of generality one can assume that $(r, x_1) \in M$ and $(r, x_2) \in N$ for some $r \in Y$. Because of (R5) it follows from the former that either $x_1 <_1 r \wedge r <_2 x_1$ or $r <_1 x_1 \wedge x_1 <_2 r$, and that there exists no $w \in Y$ such that w is larger than both, x_1 and r, with respect to $<_1$ and $<_2$. Then also $x_1 <_1 r \wedge r <_2 x_1$ or $r <_1 x_1 \wedge x_1 <_2 r$ due to (18) and (19). Thus, from $(r, x_2) \in N$, it follows that there is also no $w \in Y$ such that (R3) is violated.

Now suppose that $r <_1 x_1 \wedge x_1 <_2 r$. Then to satisfy restraints (R3) and (R6) it needs to be $x_2 <_1 x_1 \wedge x_1 <_2 x_2$ for the amalgam, and furthermore $(x_1, x_2), (x_2, x_1) \in M$.

Assumption 1: Suppose that there exists $w \in Y$ such that $x_1 <_1 w$, and $x_1 <_2 w$. Then by (18) and (19) it is $x_2 <_1 w$, and $x_2 <_2 w$. Therefore, it is $r <_1 w$, and, hence, $r <_2 w$ so that (R3) is satisfied in $\mathcal{X}_2$. But then (R5) is violated in $\mathcal{X}_1$.

Assumption 2: Suppose that there exists $w \in Y$ such that $x_1 <_1 w$, $w <_2 x_1$, and $(x_2, w) \in N$. Then as a result of restraint (R6) in $\mathcal{X}_2$ it is $(r, w) \in N$. But this would imply a violation of (R6) in $\mathcal{X}_1$.

Hence, none of the assumptions can hold. Therefore, $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate with $x_2 <_1 x_1$, $x_1 <_2 x_2$, and $(x_1, x_2), (x_2, x_1) \in M$.

On the other hand, if $x_1 <_1 r \wedge r <_2 x_1$, analogous reasoning applies when exchanging the roles of $<_1$ and $<_2$. Therefore, in Case 4 there always exists a structure in $\mathscr{X}$ that functions as an amalgam of $\mathcal{X}_1$ and $\mathcal{X}_2$.

Hence, $\mathscr{X}$ has the amalgamation property, and therefore, is an amalgamation class. □

▷ Lemma 3.8
Av(123, 132) is isomorphic to the class $\mathscr{X}_0$ of all $\{<_1, <_2\}$-reducts of structures in the class $\mathscr{X}$ as defined in Theorem 3.7.

Proof. Let $\mathcal{X} \in \mathscr{X}_0$. Then by the definition of $\mathscr{X}$, $\mathcal{X}$ avoids the patterns 123 and 132. Therefore, $\mathscr{X}_0 \subseteq \mathrm{Av}(123, 132)$.

Let $\pi \in \mathrm{Av}(123, 132)$ be an n-permutation. By Lemma 3.6, π is of the form

$$\pi = \pi_1 \ominus \ldots \ominus \pi_k$$

for some $k \in \mathbb{N}$ with π_i of the form

$$\pi_i = id_{n_i}^r \oplus 1,$$

for all $i \in \{1, \ldots k\}$, where $n_1, \ldots, n_k \in \mathbb{N}_0$, and $\sum_{i=1}^k (n_i + 1) = n$.

Therefore, for each $i \in \{1, \ldots, k\}$, $\pi_i = (n_i, \ldots, 1, n_i + 1)$. Consider the $\{<_1, <_2, D, U, M, N\}$-structure $\mathcal{X}_i$ with domain $[(n_i + 1)]$ and for $x, y \in \mathrm{dom}\,(\mathcal{X}_i)$

$$\begin{aligned}
x <_1^{\mathcal{X}_i} y &\text{ if and only if } x <_1^{\pi_i} y,\\
y <_2^{\mathcal{X}_i} y &\text{ if and only if } x <_2^{\pi_i} y,\\
x \in D^{\mathcal{X}_i} &\text{ if and only if } x \in [n_i],\\
x \in U^{\mathcal{X}_i} &\text{ if and only if } x = n_i + 1,\\
(x, y) \in M^{\mathcal{X}_i} &\text{ if and only if } x = y \wedge x = n_i + 1,\\
(y, x) \in N^{\mathcal{X}_i} &\text{ if and only if } x < n_i + 1 \vee y < n_i + 1.
\end{aligned}$$

Then it is easy to confirm that $\mathcal{X}_i \in \mathscr{X}$, and that π_i is isomorphic to the $\{<_1, <_2\}$-reduct of $\mathcal{X}_i$. Next, consider the $\{<_1, <_2, D, U, M, N\}$-structure $\mathcal{X}$ with domain $X = \biguplus_{i=1}^k \mathrm{dom}\,(\mathcal{X}_i)$ and

$$<_1^{\mathcal{X}} := \left(\biguplus_{i=1}^k <_1^{\mathcal{X}_i} \right) \uplus \left(\biguplus_{\substack{i,j \in [k] \\ i < j}} X_i \times X_j \right) \qquad <_2^{\mathcal{X}} := \left(\biguplus_{i=1}^k <_2^{\mathcal{X}_i} \right) \uplus \left(\biguplus_{\substack{i,j \in [k] \\ i < j}} X_j \times X_i \right)$$

$$D^{\mathcal{X}} := \biguplus_{i=1}^k D^{\mathcal{X}_i} \qquad U^{\mathcal{X}} := \biguplus_{i=1}^k U^{\mathcal{X}_i}$$

$$M^{\mathcal{X}} := \left(\biguplus_{i=1}^k M^{\mathcal{X}_i} \right) \uplus \left(\biguplus_{\substack{i,j \in [k] \\ i \neq j}} X_i \times X_j \right) \qquad N^{\mathcal{X}} := \biguplus_{i=1}^k N^{\mathcal{X}_i}$$

Then, analogously to the reasoning used in the proof of Theorem 3.7, it follows that $\mathcal{X} \in \mathscr{X}$. Because of the form of π and the definition of $\mathcal{X}$, the $\{<_1, <_2\}$-reduct of $\mathcal{X}$ is isomorphic to π. Hence, π is isomorphic to a structure in $\mathscr{X}_0$, and thus, $\mathrm{Av}(123, 132) \subseteq \mathscr{X}_0$.

Combining both parts yields the above statement. That is, $\mathrm{Av}(123, 132)$ is indeed isomorphic to the class $\mathcal{X}_0$ of $\{<_1, <_2\}$-reducts of elements of $\mathscr{X}$. □

Since $\mathscr{X}$ is an amalgamation class, it does have a Fraïssé limit. Let $\mathcal{D}$ be the Fraïssé limit of

$\mathscr{X}$. Then $\mathcal{D}$ is homogeneous by Theorem 2.30, and therefore ω-categorical by Lemma 2.35. Therefore, the $\{<_1, <_2\}$-reduct $\mathcal{D}_0$ of $\mathcal{D}$ is ω-categorical by Lemma 2.36.

Since $\mathcal{D}$ is the Fraïssé limit of $\mathscr{X}$, and therefore all structures in $\mathscr{X}$ can be embedded into $\mathcal{D}$, due to Lemma 3.8, each permutation avoiding the patterns 123 and 132 can be embedded into $\mathcal{D}_0$. By Theorem 2.40 and since $\mathcal{D}_0$ is countably infinite and ω-categorical, $\mathrm{Av}(123, 132)$ therefore has an ω-categorical model companion.

3.3.2 Class Av(123, 231)

Before taking any look at the model theoretic properties it is best to get a better understanding of the structure of permutations in this class. This is achieved by the following lemma.

▻ **Lemma 3.9**
Let $\pi \in \mathrm{Av}(123, 231)$ be a permutation of length n. Then π is of the form

$$\pi = \rho_1 \ominus (\rho_2 \oplus \rho_3)$$

where ρ_1, ρ_2, ρ_3 are reversals of identity permutations of lengths $n_1, n_2, n_3 \in \mathbb{N}_0$, that is,

$$\rho_i = (n_i, n_i - 1, \ldots, 2, 1) = id^r_{n_i} \tag{20}$$

if ρ_i is of length $n_i > 0$. For $n_i = 0$, ρ_i is taken to be empty, marked by $\rho_i = \emptyset$.

Proof. Note that $n = n_1 + n_2 + n_3$ has to hold as direct and skew sum result in a permutation of total length of the permutations added up.

If $n = 1$, then the above statement is rather obvious, since choosing $\rho_1 = 1, \rho_2 = \emptyset, \rho_3 = \emptyset$ satisfies equation (20):

$$1 = \pi = 1 \ominus (\emptyset \oplus \emptyset) = 1 \ominus \emptyset = 1.$$

So it can be assumed for the rest of the proof that π has at least length 2.

Since $\pi = p_1 \cdots p_n$ has finite length and its entries are pairwise distinct, there exists a unique smallest entry p_j. Consider the following three cases.

1. $j = n$
2. $j = 1$
3. $1 < j < n$

Note that the third case only applies to $n \geq 3$.

Case 1: $j = n$. Since p_j is the smallest entry, and π avoids the pattern 231, no two of the previous entries can be in increasing order. Hence, the entries of the permutation are strictly decreasing, that is,

$$p_1 > p_2 > \ldots > p_n.$$

Therefore, π itself is the reversal of the identity permutation of length n, and

$$\pi = (n, n-1, \dots, 1) = (n, n-1, \dots, 1) \ominus \emptyset = (n, n-1, \dots, 1) \ominus (\emptyset \oplus \emptyset).$$

Hence, choosing

$$\rho_1 = \pi, \qquad \rho_2 = \emptyset, \qquad \rho_3 = \emptyset$$

satisfies (20).

Case 2: $j = 1$. In this case all subsequent entries have larger values. Since π avoids the pattern 123, none of them can be in increasing order. Thus, π is of the form

$$\pi = (1, n, n-1, \dots, 2) = 1 \oplus (n-1, \dots, 1) = \emptyset \ominus (1 \oplus (n-1, \dots, 1)),$$

and by choosing

$$\rho_1 = \emptyset, \qquad \rho_2 = 1, \qquad \rho_3 = (n-1, \dots, 1),$$

equation (20) will hold.

Case 3: $1 < j < n$. By the same arguments as in Cases 1 and 2, the entries preceding p_j have to form a strictly decreasing sequence, and so have the subsequent entries. Suppose that not all neighbouring entries in the second part of π are consecutive natural numbers. That is, there exists $k \in \{j+1, \dots, n-1\}$ such that $p_{k+1} < p_k - 1$.

Since all natural numbers up to n appear exactly once in π and both parts preceding and succeding p_j form decreasing sequences, this means, that there has to be an index $m \in \{1, \dots, j-1\}$ such that $p_m = p_k - 1$. But then, π would contain the pattern 231 as the subsequence (p_m, p_k, p_{k+1}) satisfies $p_{k+1} < p_m < p_k$ and is therefore isomorphic to said pattern. Thus, it is $p_{k+1} = p_k - 1$ for all $k \in \{j+1, \dots, n-1\}$.

So if at all, the first part may be split up into two decreasing sequences that have in themselves the same form, that is, $p_{k+1} = p_k - 1$ also holds for all $k \in \{1, \dots, j-1\} \setminus \{t\}$ for some $t \in \{1, \dots, j\}$. Therefore, π is of the form

$$\pi = (n, \dots, r+1, s, \dots, 1, r, \dots, s+1)$$

for some $r, s \in [n]$, $r > s$. Thus,

$$\pi = (n-r, \dots, 1) \ominus ((s, \dots, 1) \oplus (r-s, \dots, 1)).$$

So equation (20) can be satisfied by choosing lengths

$$n_1 = n - r, \qquad n_2 = s, \qquad n_3 = r - s$$

for the reversed identity permutations ρ_1, ρ_2, ρ_3. □

With this lemma in mind it is far easier to proof that the permutation pattern avoidance class Av(123, 231) has the joint embedding property.

▻ **Lemma 3.10**
Av(123, 231) has the joint embedding property.

Proof. Let $\pi, \phi \in \mathrm{Av}(123, 231)$. Then by Lemma 3.9, they have the form

$$\pi = id^r_{n_1} \ominus (id^r_{n_2} \oplus id^r_{n_3})$$
$$\phi = id^r_{m_1} \ominus (id^r_{m_2} \oplus id^r_{m_3})$$

for some $n_1, n_2, n_3, m_1, m_2, m_3 \in \mathbb{N}$. Now consider the structure

$$\begin{aligned}\psi &:= (id^r_{n_1} \ominus id^r_{m_1}) \ominus ((id^r_{n_2} \ominus id^r_{m_2}) \oplus (id^r_{n_3} \ominus id^r_{m_3})) \\ &= id^r_{n_1+m_1} \ominus (id^r_{n_2+m_2} \oplus id^r_{n_3+m_3}).\end{aligned}$$

Obviously, both π and ϕ embed into ψ. It is also rather straightforward to see that ψ avoids both 123 and 231. Therefore, $\psi \in \mathrm{Av}(123, 231)$. Hence, Av(123, 231) does indeed have the joint embedding property. □

Now take a look at the following theorem. It directly establishes that the complete first-order theory of Av(123, 231) has an ω-categorical model companion.

▻ **Theorem 3.11**
Let $L = (<_1, <_2, P, N, Z)$ be a relational signature where $<_1, <_2$ are binary, and P, N, Z unary. Let $\mathcal{A}$ be an L-structure with domain

$$A = \mathbb{Q} \setminus \{-1, 1\},$$

and the relations defined such that

$$\begin{aligned}x \in P &\text{ if and only if } 1 < x, \\ x \in N &\text{ if and only if } x < -1, \\ x \in Z &\text{ if and only if } -1 < x \wedge x < 1, \\ x <_1 y &\text{ if and only if } x < y, \\ x <_2 y &\text{ if and only if } (1 < x \wedge y < x) \vee (-1 < y \wedge y < x \wedge x < 1) \\ &\qquad \vee (x < -1 \wedge y < x) \vee (x < -1 \wedge -1 < y \wedge y < 1).\end{aligned}$$

Then $\mathcal{A}$ is homogeneous, and the age of its $\{<_1, <_2\}$-reduct $\mathcal{A}_0$ is isomorphic to Av(123, 231). Furthermore, $\mathcal{A}_0$ is ω-categorical and the model companion of Th (Av(123, 231)).

Proof. First, note that P, N, and Z form a partition of A. This will be important throughout the proof, but will not always be explicitly stated.

The proof will be split up into three parts:

1. $\mathcal{A}$ is homogeneous.
2. $\text{Age}(\mathcal{A}_0) \cong \text{Av}(123, 231)$.
3. $\mathcal{A}_0$ is ω-categorical, and its complete first-order theory is the model-companion of $\text{Th}(\text{Av}(123, 231))$.

1. $\mathcal{A}$ is homogeneous. Let S, T be finite subsets of A, and $\mathcal{S}$, $\mathcal{T}$ substructures of $\mathcal{A}$ induced by these sets. Suppose that there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{T}$. It needs to be proven now that ι can be extended to an automorphism on $\mathcal{A}$.

Consider the restrictions of ι to P, N, and Z,

$$\iota_P = \iota\big|_{P\cap S} \qquad \iota_N = \iota\big|_{N\cap S} \qquad \iota_Z = \iota\big|_{Z\cap S}$$

Since ι is an isomorphism, the images of ι_P, ι_N, and ι_Z are solely in P, N, and Z, respectively.

Because of $(\mathbb{Q}, <)$ being a homogeneous structure, it is possible to extend ι_P, ι_N, and ι_Z to automorphisms η_P, η_N, and η_Z on P, N, and Z, respectively. Then the homomorphism $\eta : \mathcal{A} \to \mathcal{A}$ defined by

$$\eta(x) = \begin{cases} \eta_P(x), & \text{if } x \in P \\ \eta_N(x), & \text{if } x \in N \\ \eta_Z(x), & \text{if } x \in Z \end{cases}$$

is an automorphism that extends ι.

Therefore, any isomorphism between finite substructures of $\mathcal{A}$ can be extended to an automorphism on $\mathcal{A}$. So $\mathcal{A}$ is homogeneous.

2. $\text{Age}(\mathcal{A}_0) \cong \text{Av}(123, 231)$. This part really splits up into two smaller statements. Namely, it must be shown that every structure in $\text{Age}(\mathcal{A}_0)$ avoids the patterns 123 and 231, and that for every permutation in $\text{Av}(123, 231)$ there exists a structure in $\text{Age}(\mathcal{A}_0)$ isomorphic to it.

For the first part it helps to look more closely at what figuratively are descents and ascents in the structures in $\text{Age}(\mathcal{A}_0)$. Let $\mathcal{X} \in \text{Age}(\mathcal{A}_0)$. Consider two elements $x, y \in \text{dom}(\mathcal{X})$ of this structure with $x <_1 y$ and $x <_2 y$. From the former it follows that $x < y$. In combination with the latter this means that $x < -1$ and $-1 < y < 1$. Therefore, it has to be $x \in N$, and $y \in Z$.

Similarly, consider two elements $x, y \in \text{dom}(\mathcal{X})$ such that $x <_1 y$ and $y <_2 x$. Again, it follows from the former that $x < y$. Together with the latter premise this implies that either

- $y \in P$ and x in any of the sets P, N, and Z, or
- $x, y \in N$, or
- $x, y \in Z$.

Suppose now that $\mathcal{X}$ contains the pattern 123, that is, there are $r, s, t \in \mathrm{dom}(\mathcal{X})$ with $r <_1 s <_1 t$ and $r <_2 s <_2 t$. Since $r <_1 s$ and $r <_2 s$ as well as $s <_1 t$ and $s <_2 t$, it has to be $s \in Z$, and simultaneously $s \in N$. But this contradicts the fact that by the definition of the relations P and Z they are disjoint. Therefore, $\mathcal{X}$ cannot contain the pattern 123.

Suppose now that $\mathcal{X}$ contains the pattern 231, that is, there exist $u, v, w \in \mathrm{dom}(\mathcal{X})$ such that $u <_1 v <_1 w$ and $v <_2 w <_2 u$. Since $v <_1 w$ and $v <_2 w$, it is $v \in N$ and $w \in Z$. Furthermore, $u <_1 v$ and $v <_2 u$ implies $u \in N$. But on the other hand, $u <_1 w$ and $w <_2 u$ implies $u \in Z$. This is a contradiction. Therefore, $\mathcal{X}$ avoids the pattern 231.

For the second part, consider a permutation $\pi \in \mathrm{Av}(123, 231)$ of length n, which can be written in the form $\pi = p_1 p_2 \cdots p_n$. If the entries of π are strictly decreasing, π is of the form $(n, \ldots, 1)$ and isomorphic to an arbitrary substructure of $A \cap N$ with n elements, as well as to an arbitrary one of $A \cap P$ with n elements, and one of $A \cap Z$. This is due to the elements of A in any one of the subsets P, N, and Z being sorted in opposite order by $<_1$ and $<_2$.

From Lemma 3.9 it is known that in general π can be written as

$$\rho_1 \ominus (\rho_2 \oplus \rho_3)$$

with reversed identity permutations ρ_1, ρ_2, ρ_3 of lengths $n_1, n_2, n_3 \in \mathbb{N}_0$, $n_1 + n_2 + n_3 = n$. Because of the above observation, ρ_1 is isomorphic to an arbitrary substructure of $\mathcal{A}_0$ generated by a subset A_1 of $A \cap N$ with n_1 elements. In the same way, ρ_2 is isomorphic to an arbitrary substructure generated by a subset A_2 of $A \cap Z$ with n_2 elements, and ρ_3 to one generated by a subset A_3 of $A \cap P$ with n_3 elements. Due to the definition of $<_1$ and $<_2$, π is therefore isomorphic to the substructure of $\mathcal{A}_0$ generated by $A_1 \cup A_2 \cup A_2$.

Combining both statements yields that indeed $\mathrm{Age}(\mathcal{A}_0) \cong \mathrm{Av}(123, 231)$.

3. $\mathcal{A}_0$ is ω-categorical, and $\mathrm{Th}(\mathcal{A}_0)$ *is the model-companion of* $\mathrm{Th}(\mathrm{Av}(123, 231))$. Since $\mathcal{A}$ is homogeneous, it is ω-categorical by Lemma 2.35. As $\mathcal{A}_0$ is its $\{<_1, <_2\}$-reduct, $\mathcal{A}_0$ is ω-categorical as well by Lemma 2.36.

To prove that $\mathrm{Th}(\mathcal{A}_0)$ is model-complete it is sufficient by Theorem 2.39 to show that $\mathcal{A}_0$ has a homogeneous expansion by countably many existentially definable relations whose complements are existentially definable in $\mathcal{A}_0$ as well.

Since $\mathcal{A}$ is homogeneous, and $\mathcal{A}_0$ is the $\{<_1, <_2\}$-reduct of $\mathcal{A}$, $\mathcal{A}_0$ does obviously have a homogeneous expansion by relations P, N, and Z. It is easy to verify that for these relations it is:

$$\begin{aligned} x \in P &\iff \exists\, y, z : y <_1 z \wedge z <_1 x \wedge x <_2 y \wedge y <_2 z \\ x \in A \setminus P &\iff \exists\, y : (y <_1 x \wedge y <_2 x) \vee (x <_1 y \wedge x <_2 y) \end{aligned}$$

$$\begin{aligned} x \in N &\iff \exists\, y : x <_1 y \wedge x <_2 y \\ x \in A \setminus N &\iff \exists\, y, z : (y <_1 z \wedge z <_1 x \wedge x <_2 y \wedge y <_2 z) \vee (y <_1 x \wedge y <_2 x) \end{aligned}$$

$$x \in Z \iff \exists y : y <_1 x \wedge y <_2 x$$
$$x \in A \setminus Z \iff \exists y, z : (y <_1 z \wedge z <_1 x \wedge x <_2 y \wedge y <_2 z) \vee (x <_1 y \wedge x <_2 y)$$

Therefore, the relations themselves and their complements are existentially definable in $\mathcal{A}_0$. Hence, $\mathrm{Th}(\mathcal{A}_0)$ is model-complete. Since, $\mathrm{Age}(\mathcal{A}_0) \cong \mathrm{Av}(123, 231)$, this implies that $\mathrm{Th}(\mathcal{A}_0)$ is the model companion of $\mathrm{Th}(\mathrm{Av}(123, 231))$. □

3.3.3 Class Av(123, 321)

As noted previously and proved by Simion and Schmidt in [22], this permutation pattern avoidance class is finite. The following proposition together with a possible proof is the main reason for this. It is restated here since its result will not only be important for the avoidance class at hand but also some permutation pattern avoidance classes considered later on. Note also that this is a special case of the Erdös-Szekeres theorem [11].

▻ **Proposition 3.12**
Each permutation of length at least 5 contains the pattern 123 or the pattern 321.

Proof. First, note that it is sufficient to prove the assertion for permutations of length 5. This is due to any longer permutations always containing a pattern of length 5 which is of course isomorphic to a permutation of length 5. So, if the assertion holds for permutations of length 5 it also holds for longer permutations.

Let $\pi = p_1p_2p_3p_4p_5$ be an arbitrary permutation of length 5. Without loss of generality we can assume that $p_1 > p_2$. Otherwise, exchange $<$ for $>$, and vice versa, in the following implications.

As π would otherwise contain the pattern 321, we have that $p_2 < p_3$, $p_2 < p_4$, and $p_2 < p_5$, respectively. Similarly, so π avoids the pattern 123, we then have $p_3 > p_4$, and $p_4 > p_5$. But this yields $p_3 > p_4 > p_5$, so π contains the pattern 321. Since in any other case π would contain the pattern 123 or 321 as well, we now know that no permutation of length 5 is in the class $\mathrm{Av}(123, 321)$.

Therefore, each permutation that is of at least length 5 contains the pattern 123 or the pattern 321. □

From Proposition 3.12 it follows that the permutation class $\mathrm{Av}(123, 321)$ is finite, and contains only permutations up to length 4. Closer inspection yields

$$\mathrm{Av}(123, 321) = \{1, 12, 21, 132, 213, 231, 312, 2143, 2413, 3142, 3412\}.$$

The class does not have the joint embedding property since there is no permutation that contains both the pattern 2143 and the pattern 3412.

Therefore, this class will not be studied any further here. But before moving on to the next example, note the following generalisation of the propostition stated above.

▻ **Corollary 3.13**
Any permutation class that avoids the patterns 123 and 321, possibly among others, is finite.

Proof. This follows directly from Proposition 3.12 when taking the fact into account that for $F \subseteq F'$ it is $\mathrm{Av}(F') \subseteq \mathrm{Av}(F)$. □

3.3.4 Class Av(132, 213)

This is one of the Fraïssé classes as was shown by Cameron in [7] where it is referred to as the class J^*/J. The elements of this class are exactly the permutations that form a decreasing sequence of increasing subsequences.

The only other class symmetric to this one is $\mathrm{Av}(231, 312)$, which is the class J/J^* in [7]. Apart from the trivial class which avoids no patterns, these two are the only ones with the amalgamation property that avoid patterns of length 3.

Since this class does thereby have the joint embedding and amalgamation property, no further investigation will take place here.

3.3.5 Class Av(132, 312)

Note that the permutations in this class have a very peculiar structure. Since they avoid the patterns 132 and 312 it is the case for any n-permutation $\pi \in \mathrm{Av}(132, 312)$, $\pi = p_1 p_2 \cdots p_n$ that for all $i \in [n]$, p_i is either larger or smaller than all previous entries. This helps in proving that the following lemma holds.

▻ **Lemma 3.14**
$\mathrm{Av}(132, 312)$ has the joint embedding property.

Proof. Let $\pi, \phi \in \mathrm{Av}(132, 312)$. Let S be the set of indices of entries of ϕ with an index of at least 2 that are smaller than all previous entries. Now consider the $|\pi| + |\phi|$-permutation ψ defined by

$$\psi(i) = \begin{cases} \pi(i) + |S|, & \text{if } i \leq |\pi| \\ \phi(i - |\pi|), & \text{if } (i - |\pi|) \in S \\ \phi(i - |\pi|) + |\pi|, & \text{otherwise} \end{cases}$$

Obviously, ψ embeds both π and ϕ. Furthermore, ψ contains neither 132 nor 312 by construction. Hence, $\psi \in \mathrm{Av}(132, 312)$. Therefore, $\mathrm{Av}(132, 312)$ has the joint embedding property. □

To prove that the complete first-order theory of this class has an ω-categorical model companion it will be necessary to have that there exists a partition of $\mathcal{Q}^+$ into two sets that are both dense in $\mathbb{Q}^+$. The following proposition does exactly that.

▻ **Proposition 3.15**
Let $\mathbb{Q}^+$ denote the positive rational numbers. There exist $Q, R \subseteq \mathbb{Q}^+$ dense in $\mathbb{Q}^+$ such that Q and R form a partition of $\mathbb{Q}^+$, that is, $Q \cap R = \emptyset$ and $Q \cup R = \mathbb{Q}^+$.

Proof. Note that it is sufficient to give an example for sets Q and R where Q and R have the stated properties. Let

$$Q = \left\{ \frac{m}{2^n} \mid m \in \mathbb{N}, n \in \mathbb{N}_0 \right\}.$$
$$R = \mathbb{Q}^+ \setminus Q.$$

Then Q and R obviously form a partition of $\mathbb{Q}^+$.

Choose $x \in \mathbb{Q}^+$ arbitrarily. Let ϵ be a small positive rational number. It now remains to prove that there exists $y \in R$ with $x - \epsilon < y < x + \epsilon$, and $z \in Q$ with $x - \epsilon < z < x + \epsilon$.

Since

$$\lim_{n \to \infty} \frac{1}{2^n} = 0,$$

there exists $n \in \mathbb{N}$ such that

$$\frac{1}{2^n} < \epsilon.$$

Since $\mathbb{Q}$ has the Archimedian property, and since $\frac{1}{2^n} < 2\epsilon$, there exists $m \in \mathbb{N}$ such that

$$x - \epsilon < m \cdot \frac{1}{2^n} < x + \epsilon.$$

Since $\frac{m}{2^n} \in Q$, this implies that the intersection of any neighbourhood of x in $\mathbb{Q}^+$ with Q is non-empty.

Analogously, since

$$\lim_{n \to \infty} \frac{1}{n} = 0,$$

and there are infinitely many primes, there exists $p \in \mathbb{N}$ prime, $p > 2$, such that

$$\frac{1}{p} < \epsilon.$$

Then there exists $m \in \mathbb{N}$ such that

$$x - \epsilon < m \cdot \frac{1}{p} < (m+1) \cdot \frac{1}{p} < x + \epsilon.$$

The only way that both, $\frac{m}{p}$ and $\frac{m+1}{p}$, could not be in R is if $p \mid m$ and $p \mid (m+1)$. This would imply that $p \mid ((m+1)-m)$, that is $p \mid 1$. Since p is prime, this cannot hold. Therefore, at least one of $\frac{m}{p}$ and $\frac{m+1}{p}$ is in R. Hence, the intersection of any neighbourhood of x in $\mathbb{Q}^+$ with R is non-empty.

Since x and ϵ above were chosen arbitrarily, this means that Q and R are dense in $\mathbb{Q}^+$, and form a partition of $\mathbb{Q}^+$. □

Next, take a look at the following theorem. It proves neatly that the complete first-order theory of $\mathrm{Av}(132, 312)$ does indeed have an ω-categorical model companion.

▶ Theorem 3.16
Let Q, R be a partition of $\mathbb{Q}^+$ such that Q and R are both dense in $\mathbb{Q}^+$. Let $L = (<_1, <_2, P, N)$ be a relational signature where $<_1, <_2$ are binary, and P, N unary. Consider an L-structure $\mathcal{A}$ with domain

$$A = Q \cup \{x \in \mathbb{Q} \mid (-x) \in R\},$$

and the relations defined such that

$$\begin{aligned} x \in P &\text{ if and only if } 0 < x, \\ x \in N &\text{ if and only if } x < 0, \\ x <_1 y &\text{ if and only if } x < y, \\ x <_2 y &\text{ if and only if } |x| < |y|. \end{aligned}$$

Then $\mathcal{A}$ is homogeneous, and the age of its $\{<_1, <_2\}$-reduct $\mathcal{A}_0$ is isomorphic to $\mathrm{Av}(132, 312)$. Furthermore, $\mathcal{A}_0$ is ω-categorical and the model companion of $\mathrm{Th}(\mathrm{Av}(132, 312))$.

Proof. First, note that P and N form a partition of A. This will be important throughout the proof, but will not always be explicitly stated.

The proof will be split up into three parts:

1. $\mathcal{A}$ is homogeneous.
2. $\mathrm{Age}(\mathcal{A}_0) \cong \mathrm{Av}(132, 312)$.
3. $\mathcal{A}_0$ is ω-categorical, and its complete first-order theory is the model-companion of $\mathrm{Th}(\mathrm{Av}(132, 312))$.

1. $\mathcal{A}$ is homogeneous. Let S, T be finite subsets of A, and $\mathcal{S}$, $\mathcal{T}$ substructures of $\mathcal{A}$ induced by these sets. Suppose that there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{T}$. It needs to be proven now that ι can be extended to an automorphism on $\mathcal{A}$.

Consider the restrictions of ι to P and N,

$$\iota_P = \iota\big|_{P \cap S} \qquad\qquad \iota_N = \iota\big|_{N \cap S}$$

Since ι is an isomorphism, the images of ι_P and ι_N are solely in P and N, respectively.

Because of $(\mathbb{Q},<)$ being a homogeneous structure, $\mathbb{Q}^+$ is also homogeneous. And since Q and R are dense in $\mathbb{Q}^+$, $(Q,<)$ and $(R,<)$ are homogeneous structures as well. Hence, it is possible to extend ι_P and ι_N to automorphisms η_P and η_N on P and N, respectively. Then the homomorphism $\eta : \mathcal{A} \to \mathcal{A}$ defined by

$$\eta(x) = \begin{cases} \eta_P(x), & \text{if } x \in P \\ \eta_N(x), & \text{if } x \in N \end{cases}$$

is an automorphism that extends ι.

Therefore, any isomorphism between finite substructures of $\mathcal{A}$ can be extended to an automorphism on $\mathcal{A}$. So $\mathcal{A}$ is homogeneous.

2. $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(132, 312)$. This part really splits up into two smaller statements. Namely, it is necessary to show that every structure in $\mathrm{Age}\,(\mathcal{A}_0)$ avoids the patterns 132 and 312, and that for every permutation in $\mathrm{Av}(132, 312)$ there exists a structure in $\mathrm{Age}\,(\mathcal{A}_0)$ isomorphic to it.

For the first part it helps to look more closely at what, figuratively speaking, are descents and ascents in the structures in $\mathrm{Age}\,(\mathcal{A}_0)$. Let $\mathcal{X} \in \mathrm{Age}\,(\mathcal{A}_0)$. Consider two elements $x, y \in \mathrm{dom}\,(\mathcal{X})$ of this structure with $x <_1 y$ and $x <_2 y$. From the former it follows that $x < y$. In combination with the latter this means that either $x > 0$ and $y > 0$, or $x < 0$ and $y > 0$. Therefore, it has to be $y \in P$.

Similarly, consider two elements $x, y \in \mathrm{dom}\,(\mathcal{X})$ such that $x <_1 y$ and $y <_2 x$. Again, it follows from the former that $x < y$. Together with the latter premise this implies that either both are negative, or $x < 0$ and $y > 0$. Therefore, $x \in N$ holds in this case.

Suppose now that $\mathcal{X}$ contains the pattern 132, that is, there exist $u, v, w \in \mathrm{dom}\,(\mathcal{X})$ with $u <_1 v <_1 w$ and $u <_2 w <_2 v$. From the first observation it follows that $v \in P$, while it follows from the second that $v \in N$. Since this is a contradiction, $\mathcal{X}$ cannot contain the pattern 132.

Conversely, suppose that $\mathcal{X}$ contains the pattern 312, that is, there exist $u, v, w \in \mathrm{dom}\,(\mathcal{X})$ with $u <_1 v <_1 w$ and $w <_2 u <_2 v$. Then it follows from the first observation that $v \in P$, and from the second that $v \in N$. Once again, this is a contradiction, so $\mathcal{X}$ avoids the pattern 312.

For the second part, consider a permutation $\pi \in \mathrm{Av}(132, 312)$ of length n, which can be written in the form $\pi = p_1 p_2 \cdots p_n$. Next, find $x_1, \ldots, x_n \in \mathbb{Q}^+$ with

$$x_1 < x_2 < \ldots < x_n,$$
$$x_i \in \begin{cases} Q, & \text{if for all } j = 1, \ldots, i-1\text{: } p_j < p_i \\ R, & \text{otherwise} \end{cases}$$

for all $i \in \{1, \ldots, n\}$. This is possible since Q and R are dense in $\mathbb{Q}^+$. Let $B = \{x_1, \ldots, x_n\}$. Then define

$$X = (B \cap Q) \cup \{x \mid (-x) \in B \cap R\}.$$

By definition, $X \subseteq A$. Let $\mathcal{X}$ be the substructure of $\mathcal{A}$ generated by X. Then for $x, y \in X$ it is

$$\begin{aligned}
x \in P &\Leftrightarrow 0 < x \Leftrightarrow x \in B \cap Q \\
&\Leftrightarrow \exists\, i \in \{1, \ldots, n\} : x = x_i \wedge (\forall\, j \in \{1, \ldots i-1\} : p_j < p_i) \\
x \in N &\Leftrightarrow x < 0 \Leftrightarrow (-x) \in B \cap R \\
&\Leftrightarrow \exists\, i \in \{1, \ldots, n\} : -x = x_i \wedge (\exists\, j \in \{1, \ldots, i-1\} : p_i < p_j) \\
x <_1 y &\Leftrightarrow x < y \\
&\Leftrightarrow (x, y \in B \cap Q \wedge x < y) \vee ((-x), (-y) \in B \cap R \wedge -y < -x) \\
&\qquad \vee ((-x) \in B \cap R \wedge y \in Q \cap R) \\
x <_2 y &\Leftrightarrow |x| < |y| \Leftrightarrow \exists\, i, j : i < j \wedge |x| = x_i \wedge |y| = x_j
\end{aligned}$$

Compairing this to the properties of π yields that $\mathcal{X}$ is isomorphic to the permutation one started the contruction with.

Combining both statements yields that indeed $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(132, 312)$.

3. $\mathcal{A}_0$ is ω-categorical, and $\mathrm{Th}\,(\mathcal{A}_0)$ *is the model-companion of* $\mathrm{Th}\,(\mathrm{Av}(132, 312))$. Since $\mathcal{A}$ is homogeneous, it is ω-categorical by Lemma 2.35. As $\mathcal{A}_0$ is its $\{<_1, <_2\}$-reduct, $\mathcal{A}_0$ is ω-categorical as well by Lemma 2.36.

To prove that $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete it is sufficient by Theorem 2.39 to show that $\mathcal{A}_0$ has a homogeneous expansion by countably many existentially definable relations whose complements are existentially definable in $\mathcal{A}_0$ as well.

Since $\mathcal{A}$ is homogeneous, and $\mathcal{A}_0$ is the $\{<_1, <_2\}$-reduct of $\mathcal{A}$, $\mathcal{A}_0$ does obviously have a homogeneous expansion by relations P and N. It is easy to verify that for these relations it is:

$$\begin{aligned}
x \in P &\Longleftrightarrow \exists\, y : y <_1 x \wedge y <_2 x \\
x \in N &\Longleftrightarrow \exists\, y : x <_1 y \wedge y <_2 x
\end{aligned}$$

Note that $N = A \setminus P$. Therefore, the relations themselves and their complements are existentially definable in $\mathcal{A}_0$. Hence, $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete. Since $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(132, 312)$, this implies that $\mathrm{Th}\,(\mathcal{A}_0)$ is the model companion of $\mathrm{Th}\,(\mathrm{Av}(132, 312))$. □

3.4 Avoiding three patterns

When avoiding three patterns of length 3 there are once again five sets of symmetric sets of patterns. They are the following. A detailed proof of this can be found in [22].

- $\{123, 132, 213\}$, $\{231, 312, 321\}$

- $\{123, 132, 231\}$, $\{213, 231, 321\}$, $\{123, 213, 312\}$, $\{132, 312, 321\}$, $\{132, 231, 321\}$, $\{213, 312, 321\}$, $\{123, 213, 231\}$, $\{123, 132, 312\}$
- $\{123, 132, 321\}$, $\{123, 231, 321\}$, $\{123, 213, 321\}$, $\{123, 312, 321\}$
- $\{123, 231, 312\}$, $\{132, 213, 321\}$
- $\{132, 213, 231\}$, $\{213, 231, 312\}$, $\{132, 213, 312\}$, $\{132, 231, 312\}$

There are just three Wilf-equivalence classes in this case [22]. Basically, the third class is finite, the first is enumerated by the Fibonacci numbers, and the others have n permutations of length n for all $n \in \mathbb{N}$.

The classes considered in the following sections are the ones that avoid the sets of patterns $\{123, 132, 213\}$, $\{123, 132, 231\}$, $\{123, 132, 321\}$, $\{123, 231, 312\}$ and $\{132, 213, 231\}$, respectively.

3.4.1 Class Av(123, 132, 213)

First note that the following holds for this class.

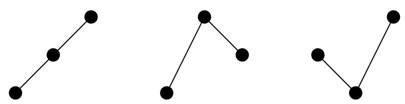
▻ Lemma 3.17
$\mathrm{Av}(123, 132, 213)$ has the joint embedding property.

Proof. Obviously, the patterns 123, 132 and 213 are $\ominus$-indecomposable. So by Corollary 2.18 in conjunction with Corollary 2.16 $\mathrm{Av}(123, 132, 213)$ has the joint embedding property. □

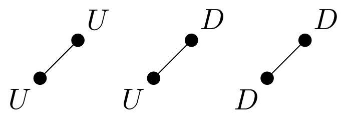
Now take a look at the class of structures defined in the following theorem. Afterwards, it will be proven that the $\{<_1, <_2\}$-reducts of the structures in the following class are exactly the structures in $\mathrm{Av}(123, 132, 213)$.

▻ Theorem 3.18
Let $\mathscr{X}$ be the class of all $(U, D, <_1, <_2)$-structures $\mathcal{X}$ that avoid

(R1)



(R2)



with U, D unary relations, and $<_1$, $<_2$ binary relations for which

- $U \cap D = \emptyset$, $U \cup D = \mathrm{dom}(\mathcal{X})$,
- $<_1$, $<_2$ are strict total orders.

Then $\mathcal{X}$ is an amalgamation class.

Proof. To prove that $\mathscr{X}$ is an amalgamation class, it is necessary and sufficient by Theorem 2.30 to show that $\mathscr{X}$ is closed under isomorphism, and has the hereditary property, joint embedding property, and amalgamation property.

Since the structures in $\mathscr{X}$ are defined by avoiding certain substructures, the class is closed under isomorphism, and does have the hereditary property.

It is not as easy to see that the joint embedding property holds. Let $\mathcal{X}, \mathcal{Y} \in \mathscr{X}$ with $X = \mathrm{dom}(\mathcal{X})$, $Y = \mathrm{dom}(\mathcal{Y})$. Without loss of generality it can be assumed that X and Y are disjoint. If this would not be the case and they share an element x, mark this element with 1 in X and with 2 in Y. This does not change anything about the properties of the elements of each respective set.

Let $\mathcal{Z}$ be the L-structure defined on the domain $Z = X \cup Y$ with interpretations of the relations such that for $u, v \in Z$

$$\begin{aligned}
<_1^{\mathcal{Z}} &:= <_1^{\mathcal{X}} \cup <_1^{\mathcal{Y}} \cup (X \times Y),\\
<_2^{\mathcal{Z}} &:= <_2^{\mathcal{X}} \cup <_2^{\mathcal{Y}} \cup (Y \times X),\\
D^{\mathcal{Z}} &:= D^{\mathcal{X}} \cup D^{\mathcal{Y}},\\
U^{\mathcal{Z}} &:= U^{\mathcal{X}} \cup U^{\mathcal{Y}}.
\end{aligned}$$

Obviously, $\mathcal{Z}$ embeds both, $\mathcal{X}$ and $\mathcal{Y}$. To prove that $\mathcal{Z} \in \mathscr{X}$ holds as well, it needs to be verified that $\mathcal{Z}$ satisfies (R1) and (R2), and the interpretations of the relations have the stated properties.

By the given definition, $<_1^{\mathcal{Z}}$ and $<_2^{\mathcal{Z}}$ are strict total orders, and

$$\begin{aligned}
D^{\mathcal{Z}} \cap U^{\mathcal{Z}} &= (D^{\mathcal{X}} \cup D^{\mathcal{Y}}) \cap (U^{\mathcal{X}} \cup U^{\mathcal{Y}})\\
&= (D^{\mathcal{X}} \cap U^{\mathcal{X}}) \cup (D^{\mathcal{X}} \cap U^{\mathcal{Y}}) \cup (D^{\mathcal{Y}} \cap U^{\mathcal{X}}) \cup (D^{\mathcal{Y}} \cap U^{\mathcal{Y}})\\
&= \emptyset \cap \emptyset \cap \emptyset \cap \emptyset = \emptyset,\\
D^{\mathcal{Z}} \cup U^{\mathcal{Z}} &= (D^{\mathcal{X}} \cup D^{\mathcal{Y}}) \cup (U^{\mathcal{X}} \cup U^{\mathcal{Y}})\\
&= (D^{\mathcal{X}} \cup U^{\mathcal{X}}) \cup (D^{\mathcal{Y}} \cup U^{\mathcal{Y}})\\
&= X \cup Y = Z.
\end{aligned}$$

Furthermore, for any $x \in X, y \in Y$ it is $x <_1 y$ and $y <_2 x$. Hence, if $\mathcal{Z}$ would violate (R1) or (R2), one of $\mathcal{X}$ and $\mathcal{Y}$ would violate this restraint as well. Since $\mathcal{X}, \mathcal{Y} \in \mathscr{X}$ this implies that $\mathcal{Z}$ satisfies both restraints. Therefore, $\mathscr{X}$ has the joint embedding property.

The remaining property that needs to be verified is the amalgamation property. It is sufficient to consider two-point-amalgamation [8]. Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y} \in \mathscr{X}$ such that $\mathcal{Y}$ is a substructure of $\mathcal{X}_1$ as well as $\mathcal{X}_2$, and

$$\begin{aligned}
X_1 &= \mathrm{dom}(\mathcal{X}_1), & X_2 &= \mathrm{dom}(\mathcal{X}_2), & Y &= \mathrm{dom}(\mathcal{Y}),\\
X_1 &= Y \cup \{x_1\}, & X_2 &= Y \cup \{x_2\}.
\end{aligned}$$

In the same way as in the proof of Theorem 3.7 it suffices to prove this for the following four cases:

Case 1: $\exists\, x : x_1 <_1 x <_1 x_2$
$\exists\, y : x_1 <_2 y <_2 x_2$

Case 2: $\exists\, x : x_1 <_1 x <_1 x_2$
$\exists\, y : x_2 <_2 y <_2 x_1$

Case 3: $\exists\, x : x_1 <_1 x <_1 x_2$
$\forall\, y : (y <_2 x_1 \wedge y <_2 x_2) \vee (x_1 <_2 y \wedge x_2 <_2 y)$

Case 4: $\forall\, x : (x <_1 x_1 \wedge x <_1 x_2) \vee (x_1 <_1 x \wedge x_2 <_1 x)$
$\forall\, y : (y <_2 x_1 \wedge y <_2 x_2) \vee (x_1 <_2 y \wedge x_2 <_2 y)$

Case 1:

$$\exists\, x : x_1 <_1 x <_1 x_2 \tag{3}$$
$$\exists\, y : x_1 <_2 y <_2 x_2 \tag{4}$$

In this case it is impossible to amalgamate $\mathcal{X}_1$ and $\mathcal{X}_2$ as due to (3) and (4) any amalgam would violate (R1).

Assuming that this case actually occurs yields the following. Either one of x or y lies inbetween x_1 and x_2, that is, $x_1 <_1 y <_1 x_2$ or $x_1 <_2 x <_2 x_2$, or both do not.

Suppose that the former holds. Without loss of generality it can be assumed that $x_1 <_2 x <_2 x_2$. Then for (R2) to be satisfied in $\mathcal{X}_1$ it would have to be $x \in U$. On the other hand, for it to be satisfied in $\mathcal{X}_2$ it would have to be $x \in D$. Since this is a contradiction, our assumption is falsified.

Now suppose that neither x nor y lies inbetween x_1 and x_2. Then for x it is either $x <_2 x_1$ or $x_2 <_2 x$, and for y it is either $y <_1 x_1$ or $x_2 <_1 y$. If $x <_2 x_1$ then both $y <_1 x_1$ and $x_2 <_1 y$ result in a violaton of (R2) in $\mathcal{X}_2$. On the other hand, if $x_2 <_2 x$, both $y <_1 x_1$ and $x_2 <_1 y$ result in a violation of (R2) in $\mathcal{X}_1$. Therefore, this cannot be the case either.

As both cases yield direct contradiction within $\mathcal{X}_1$ or $\mathcal{X}_2$ they cannot hold. Thus, this case never actually applies.

Case 2:

$$\exists\, x : x_1 <_1 x <_1 x_2 \tag{5}$$
$$\exists\, y : x_2 <_2 y <_2 x_1 \tag{6}$$

There is only one possible amalgam which is gained by adding $x_1 <_1 x_2$ and $x_2 <_2 x_1$. Due to its structure this amalgam satisfies restraint (R1).

The only way (R2) could be violated is if there exists $z \in Y$ such that either $z <_1 x_1$ and $z <_2 x_2$, or $x_2 <_1 z$ and $x_1 <_2 z$. Suppose the former of these two possibilities applies. Then z, x and x_2 would form a substructure of $\mathcal{X}_2$. Because of (5), $z <_1 x <_1 x_2$. But then no matter how x was positioned with respect to $<_2$, this substructure would contain a pattern from (R1). Analogously, a contradiction is obtained when considering the substructure of $\mathcal{X}_2$ formed by z, x, and x_x if $z \in Y$ such that $x_2 <_1 z$ and $x_1 <_2 z$. Thus, $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate in this case.

Case 3:

$$\exists\, x : x_1 <_1 x <_1 x_2 \tag{7}$$
$$\forall\, y : (y <_2 x_1 \wedge y <_2 x_2) \vee (x_1 <_2 y \wedge x_2 <_2 y) \tag{8}$$

In this case $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate with $x_1 <_1 x_2$ due to (7) and $x_2 <_2 x_1$. Similarly to the situation in Case 2, the latter makes sure that the amalgam is indeed admissible. The only way it could violate one of the restraints is if there exists $z \in Y$ such that either $z <_1 x_1$ and $z <_2 x_2$, or $x_2 <_1 z$ and $x_1 <_2 z$. In both cases the same reasoning given above yields a contradiction. Hence, $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate in Case 3.

Case 4:

$$\forall\, x : (x <_1 x_1 \wedge x <_1 x_2) \vee (x_1 <_1 x \wedge x_2 <_1 x) \tag{9}$$
$$\forall\, y : (y <_2 x_1 \wedge y <_2 x_2) \vee (x_1 <_2 y \wedge x_2 <_2 y) \tag{10}$$

Suppose that x_1 and x_2 are both either in U or D, respectively. Then $\mathcal{X}_1$ and $\mathcal{X}_2$ can be amalgamated by identifying x_1 and x_2. On the other hand, if one of x_1 and x_2 is in D and the other in U it is known from the restraints being satisfied in $\mathcal{X}_1$ as well as $\mathcal{X}_2$ that there exists no $z \in Y$ for which $z <_1 x_1$ and simultaneously $z <_2 x_1$, and none for which at the same time $x_1 <_1 z$ and $x_1 <_2 z$. Therefore, $\mathcal{X}_1$ and $\mathcal{X}_2$ can be amalgamated by adding $x_1 <_1 x_2$ and $x_2 <_2 x_1$.

Since $\mathcal{X}_1$ and $\mathcal{X}_2$ amalgamate in all four cases, $\mathscr{X}$ is an amalgamation class. □

▻ **Lemma 3.19**
The class $\mathscr{X}_0$ of all $\{<_1, <_2\}$-reducts of structures in $\mathscr{X}$ is isomorphic to $\mathrm{Av}(123, 132, 213)$.

Proof. Obviously, all $\mathcal{X} \in \mathscr{X}_0$ avoid the patterns 123, 132 and 213. It remains to prove that for each permutation in $\mathrm{Av}(123, 132, 213)$ there exists an isomorphic structure in $\mathscr{X}_0$.

This can be shown by the same means as done in the proof of Lemma 3.8 combined with the construction at the beginning of the proof of Theorem 3.18.

Hence, $\mathrm{Av}(123, 132, 213) \cong \mathscr{X}_0$. □

From the above in conjunction with Theorem 2.30, Lemma 2.35, Lemma 2.36 and finally Theorem 2.40 it follows that the first-order theory of $\mathrm{Av}(123, 132, 213)$ has an ω-categorical model companion.

3.4.2 Class Av(123, 132, 231)

Once again it is quite useful to first take a look at the structure of permutation in this permutation pattern avoidance class.

▻ **Lemma 3.20**
Let $\pi \in \mathrm{Av}(123, 132, 231)$ be an n-permutation. Then π is of the form

$$\pi = id^r_{n_1} \ominus (id^r_{n_2} \oplus 1)$$

for some $n_1, n_2 \in \mathbb{N}_0$, $n_1 + n_2 + 1 = n$.

Proof. Since $\{123, 231\} \subseteq \{123, 132, 231\}$ it is $\mathrm{Av}(123, 132, 231) \subseteq \mathrm{Av}(123, 231)$. Therefore, any n-permutation $\pi \in \mathrm{Av}(123, 132, 231)$ can be written as

$$\pi = id^r_{m_1} \ominus (id^r_{m_2} \oplus id^r_{m_3})$$

with $m_1, m_2, m_3 \in \mathbb{N}_0$, $m_1 + m_2 + m_3 = n$, according to Lemma 3.9. Obviously, if $m_3 = 1$, then π has the form stated above with $n_1 = m_1$ and $n_2 = m_2$.

Suppose now that for some $\pi \in \mathrm{Av}(123, 132, 231)$, $m_3 > 1$. If $m_2 > 0$ this would imply that π contains the pattern 132, since

$$132 = 1 \oplus 21 \preceq id^r_{m_2} \oplus id^r_{m_3} \preceq id^r_{m_1} \ominus (id^r_{m_2} \oplus id^r_{m_3}) = \pi.$$

Therefore, either $m_3 < 2$ or $m_2 = 0$. If $m_2 = 0$, this yields that

$$\begin{aligned}\pi &= id^r_{m_1} \ominus (id^r_0 \oplus id^r_{m_3}) = id^r_{m_1} \ominus id^r_{m_3} = id^r_{m_1+m_3} \\ &= id^r_n = id^r_{n-1} \ominus 1 = id^r_{n-1} \ominus (\emptyset \oplus 1) \\ &= id^r_{n-1} \ominus (id^r_0 \oplus 1).\end{aligned}$$

Therefore, π has the desired form with $n_1 = n - 1$ and $n_2 = 0$. On the other hand, if $m_3 < 2$, then the only remaining case that is interesting is $m_3 = 0$. In this case it is

$$\begin{aligned}\pi &= id^r_{m_1} \ominus (id^r_{m_2} \oplus id^r_0) = id^r_{m_1} \ominus id^r_{m_2} = id^r_{m_1+m_2} \\ &= id^r_n = id^r_{n-1} \ominus 1 = id^r_{n-1} \ominus (\emptyset \oplus 1) \\ &= id^r_{n-1} \ominus (id^r_0 \oplus 1).\end{aligned}$$

So in the same way as above, π has the stated form with $n_1 = n - 1$ and $n_2 = 0$. □

Conversely, it is straightforward to see that any finite permutation of the form as in Lemma 3.20 avoids the patterns 123, 132 and 231. The fact that $\mathrm{Av}(123, 132, 231)$ has the joint embedding property can now be easily deduced from this.

▻ Lemma 3.21
$\mathrm{Av}(123, 132, 231)$ has the joint embedding property.

Proof. Let $\pi, \phi \in \mathrm{Av}(123, 132, 231)$. Then by Lemma 3.20, they have the form

$$\begin{aligned}\pi &= id^r_{n_1} \ominus (id^r_{n_2} \oplus 1) \\ \phi &= id^r_{m_1} \ominus (id^r_{m_2} \oplus 1)\end{aligned}$$

for some $n_1, n_2, m_1, m_2 \in \mathbb{N}$. Now consider the structure

$$\begin{aligned}\psi &:= (id^r_{n_1} \ominus id^r_{m_1}) \ominus ((id^r_{n_2} \ominus id^r_{m_2}) \oplus 1)\\ &= id^r_{n_1+m_1} \ominus (id^r_{n_2+m_2} \oplus 1).\end{aligned}$$

Obviously, both π and ϕ embed into ψ. It is also rather straightforward to see that ψ avoids the patterns 123, 132 and 231. Therefore, $\psi \in \mathrm{Av}(123, 132, 231)$. Hence, $\mathrm{Av}(123, 132, 231)$ does indeed have the joint embedding property. □

Now it makes sense to consider the following theorem. It will prove that the first-order theory of $\mathrm{Av}(123, 132, 231)$ has an ω-categorical model companion.

▶ Theorem 3.22
Let $L = (<_1, <_2, P, N)$ be a relational signature where $<_1, <_2$ are binary, and P, N unary. Consider an L-structure $\mathcal{A}$ with domain

$$A = \mathbb{Q},$$

and the relations defined such that

$$\begin{aligned}x \in P &\text{ if and only if } 0 < x,\\ x \in N &\text{ if and only if } x < 0,\\ x <_1 y &\text{ if and only if } x < y,\\ x <_2 y &\text{ if and only if } x \neq 0 \wedge (y = 0 \vee (y \neq 0 \wedge y < x)).\end{aligned}$$

Then $\mathcal{A}$ is homogeneous, and the age $\mathrm{Age}(\mathcal{A}_0)$ of its $\{<_1, <_2\}$-reduct $\mathcal{A}_0$ is isomorphic to $\mathrm{Av}(123, 132, 231)$.

Furthermore, $\mathcal{A}_0$ is ω-categorical and the model companion of $\mathrm{Th}(\mathrm{Av}(123, 132, 231))$.

Proof. The proof will be split up into three parts:

1. $\mathcal{A}$ is homogeneous.
2. $\mathrm{Age}(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 231)$.
3. $\mathcal{A}_0$ is ω-categorical, and its complete first-order theory is the model-companion of $\mathrm{Th}(\mathrm{Av}(123, 132, 231))$.

1. $\mathcal{A}$ is homogeneous. Let S, T be finite subsets of A, and $\mathcal{S}$, $\mathcal{T}$ substructures of $\mathcal{A}$ induced by these sets. Suppose that there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{T}$. It is now necessary to show that ι can be extended to an automorphism on $\mathcal{A}$.

Consider the restrictions of ι to P, and N,

$$\iota_P = \iota\big|_{P \cap S} \qquad\qquad \iota_N = \iota\big|_{N \cap S}$$

Since ι is an isomorphism, the images of ι_P, and ι_N are solely in P, and N, respectively. Moreover, if ι is defined on 0, $\iota(0) = 0$, since 0 is neither in N nor P.

Because of $(\mathbb{Q}, <)$ being a homogeneous structure, it is possible to extend ι_P, and ι_N to automorphisms η_P, and η_N on P, and N, respectively. Then the homomorphism $\eta : \mathcal{A} \to \mathcal{A}$ defined by

$$\eta(x) = \begin{cases} \eta_P(x), & \text{if } x \in P \\ \eta_N(x), & \text{if } x \in N \\ 0, & \text{if } x = 0 \end{cases}$$

is an automorphism that extends ι.

Therefore, any isomorphism between finite substructures of $\mathcal{A}$ can be extended to an automorphism on $\mathcal{A}$. So $\mathcal{A}$ is homogeneous.

2. Age $(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 231)$. This part really splits up into two smaller statements. Namely, it is necessary to show that every structure in Age $(\mathcal{A}_0)$ avoids the patterns 123, 132, and 231, and that for every permutation in $\mathrm{Av}(123, 132, 231)$ there exists a structure in Age $(\mathcal{A}_0)$ isomorphic to it.

For the first part it helps to look more closely at the descents and ascents in the structures in Age $(\mathcal{A}_0)$. Let $\mathcal{X} \in \text{Age}(\mathcal{A}_0)$. Consider two elements $x, y \in \text{dom}(\mathcal{X})$ of this structure with $x <_1 y$ and $x <_2 y$. From the former it follows that $x < y$. In combination with the latter this means that $y = 0$, and therefore, $x \in N$.

Similarly, consider two elements $x, y \in \text{dom}(\mathcal{X})$ such that $x <_1 y$ and $y <_2 x$. Again, it follows from the former that $x < y$. Together with the latter premise this implies that either $x, y \in P$, or $x \in P, y \in N$, or $x, y \in N$, or $y = 0$, $x \in P$.

Now suppose that $\mathcal{X}$ contains the pattern 123, that is, there exist $a, b, c \in \text{dom}(\mathcal{X})$ with $a <_1 b <_1 c$ and $a <_2 b <_2 c$. Then it follows from the first observation that $b = 0$ and $c = 0$. But this is a contradiction since $b \neq c$.

Next, suppose that $\mathcal{X}$ contains the pattern 132, that is, there exist $a, b, c \in \text{dom}(\mathcal{X})$ with $a <_1 b <_1 c$ and $a <_2 c <_2 b$. Then the first observation implies that $b = 0$ as well as $c = 0$. Again, this is a contradiction.

Finally, suppose that $\mathcal{X}$ contains the pattern 231, that is, there exist $a, b, c \in \text{dom}(\mathcal{X})$ with $a <_1 b <_1 c$ and $b <_2 c <_2 a$. Then $c = 0$. This contradicts the conclusion from the second observation that $c \neq 0$, though.

For the second part, consider a permutation $\pi \in \mathrm{Av}(123, 132, 231)$ of length n, which can be written in the form $\pi = p_1 p_2 \cdots p_n$. By Lemma 3.20 it is known that π is of the form

$$\pi = id^r_{n_1} \ominus (id^r_{n_2} \oplus 1)$$

for some $n_1, n_2 \in \mathbb{N}_0$ with $n_1 + n_2 + 1 = n$. Consider the mapping $f : [n] \to A$,

$$f(i) = \begin{cases} n+1-i, & \text{if } i \leq n_1 \\ -i, & \text{if } n_1 < i \leq n_1 + n_2 \\ 0, & \text{if } i = n \end{cases} .$$

Then $\mathrm{Im}(f)$ generates a finite substructure of $\mathcal{A}$. By the definition of $\mathcal{A}$, the $\{<_1, <_2\}$-reduct of the generated substructure is isomorphic to π.

Combining both statements yields that indeed $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 231)$.

3. $\mathcal{A}_0$ is ω-categorical, and $\mathrm{Th}\,(\mathcal{A}_0)$ *is the model-companion of* $\mathrm{Th}\,(\mathrm{Av}(123, 132, 231))$. Since $\mathcal{A}$ is homogeneous, it is ω-categorical by Lemma 2.35. As $\mathcal{A}_0$ is its $\{<_1, <_2\}$-reduct, $\mathcal{A}_0$ is ω-categorical as well by Lemma 2.36.

To prove that $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete it is sufficient by Theorem 2.39 to show that $\mathcal{A}_0$ has a homogeneous expansion by countably many existentially definable relations whose complements are existentially definable in $\mathcal{A}_0$ as well.

Since $\mathcal{A}$ is homogeneous, and $\mathcal{A}_0$ is the $\{<_1, <_2\}$-reduct of $\mathcal{A}$, $\mathcal{A}_0$ does obviously have a homogeneous expansion by relations P, and N. It is easy to verify, that for these relations it is:

$$\begin{aligned} x \in P &\iff \exists\, y, z : z <_1 y <_1 x \wedge x <_2 z <_2 y, \\ x \in A \setminus P &\iff \exists\, y : (x <_1< y \wedge x <_2 y) \vee (y <_1 x \wedge y <_2 x), \end{aligned}$$

$$\begin{aligned} x \in N &\iff \exists\, y : x <_1< y \wedge x <_2 y, \\ x \in A \setminus N &\iff \exists\, y, z : (z <_1 y <_1 x \wedge x <_2 z <_2 y) \vee (y <_1 x \wedge y <_2 x). \end{aligned}$$

Therefore, the relations themselves and their complements are existentially definable in $\mathcal{A}_0$. Hence, $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete. Since, $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 231)$, this implies that $\mathrm{Th}\,(\mathcal{A}_0)$ is the model companion of $\mathrm{Th}\,(\mathrm{Av}(123, 132, 231))$. □

3.4.3 Class Av(123, 132, 321)

By Corollary 3.13 this class is finite as it avoids both 123 and 321. Closer inspection yields that it has the following elements:

$$\mathrm{Av}(123, 132, 321) = \{1, 12, 21, 213, 231, 312, 3412\}$$

It is obviously not atomic as there is no permutation that has both permutations 213 and 231 as substructures.

3.4.4 Class Av(123, 231, 312)

Before looking at any model theoretic properties it is very helpful to get a better understanding of the inner structure of the permutations in this permutation pattern avoidance class.

▻ **Lemma 3.23**
Let $\pi \in \mathrm{Av}(123, 231, 312)$ be an n-permutation. Then π is of the form

$$\pi = id^r_{n_1} \oplus id^r_{n_2}$$

for some $n_1, n_2 \in \mathbb{N}_0$, $n_1 + n_2 = n$.

Proof. Since $\{123, 231\} \subseteq \{123, 231, 312\}$ it is $\mathrm{Av}(123, 231, 312) \subseteq \mathrm{Av}(123, 231)$. Therefore, any n-permutation $\pi \in \mathrm{Av}(123, 231, 312)$ can be written as

$$\pi = id^r_{m_1} \ominus (id^r_{m_2} \oplus id^r_{m_3})$$

with $m_1, m_2, m_3 \in \mathbb{N}_0$, $m_1 + m_2 + m_3 = n$, according to Lemma 3.9. Obviously, if $m_1 = 0$, then π has the form stated above with $n_1 = m_2$ and $n_2 = m_3$.

Suppose now that for some $\pi \in \mathrm{Av}(123, 231, 312)$, $m_1 > 0$. If $m_2 > 0$ and $m_3 > 0$ this would imply that π contains the pattern 312, since

$$312 = 1 \ominus (1 \oplus 1) \preceq id^r_{m_1} \ominus (id^r_{m_2} \oplus id^r_{m_3}) = \pi.$$

Therefore, either $m_1 = 0$ or $m_2 = 0$ or $m_3 = 0$. If $m_2 = 0$, this yields that

$$\begin{aligned}\pi &= id^r_{m_1} \ominus (id^r_0 \oplus id^r_{m_3}) = id^r_{m_1} \ominus id^r_{m_3} = id^r_{m_1+m_3} \\ &= id^r_n = id^r_n \oplus id^r_0.\end{aligned}$$

Therefore, π has the desired form with $n_1 = n$ and $n_2 = 0$. On the other hand, if $m_3 = 0$, then analogously,

$$\begin{aligned}\pi &= id^r_{m_1} \ominus (id^r_{m_2} \oplus id^r_0) = id^r_{m_1} \ominus id^r_{m_2} = id^r_{m_1+m_2} \\ &= id^r_n = id^r_n \oplus id^r_0.\end{aligned}$$

So in the same way as above, π has the stated form with $n_1 = n$ and $n_2 = 0$. □

This helps proving the following.

▻ **Lemma 3.24**
Av(123, 132, 231) has the joint embedding property.

Proof. Let $\pi, \phi \in \mathrm{Av}(123, 231, 312)$. Then by Lemma 3.23, they have the form

$$\pi = id^r_{n_1} \oplus id^r_{n_2}$$
$$\phi = id^r_{m_1} \oplus id^r_{m_2}$$

for some $n_1, n_2, m_1, m_2 \in \mathbb{N}$. Now consider the structure

$$\begin{aligned}\psi &:= (id^r_{n_1} \ominus id^r_{m_1}) \oplus (id^r_{n_2} \ominus id^r_{m_2}) \\ &= id^r_{n_1+m_1} \oplus id^r_{n_2+m_2}\end{aligned}$$

Clearly, both π and ϕ embed into ψ. It is also rather straightforward to see that ψ avoids the patterns 123, 231 and 312. Therefore, $\psi \in \mathrm{Av}(123, 231, 312)$. Hence, $\mathrm{Av}(123, 231, 312)$ has the joint embedding property. □

Now the following theorem proves that the first-order theory of the permutation pattern avoidance class has an ω-categorical model companion.

Theorem 3.25

Let $L = (<_1, <_2, P, N)$ be a relational signature where $<_1, <_2$ are binary, and P, N unary. Consider an L-structure $\mathcal{A}$ with domain

$$A = \mathbb{Q} \setminus \{0\},$$

and the relations defined such that

$$\begin{aligned} x \in P &\text{ if and only if } 0 < x, \\ x \in N &\text{ if and only if } x < 0, \\ x <_1 y &\text{ if and only if } x < y, \\ x <_2 y &\text{ if and only if } (x < 0 \wedge y < x) \vee (0 < y \wedge y < x) \vee (x < 0 \wedge 0 < y). \end{aligned}$$

Then $\mathcal{A}$ is homogeneous, and the age $\mathrm{Age}(\mathcal{A}_0)$ of its $\{<_1, <_2\}$-reduct $\mathcal{A}_0$ is isomorphic to $\mathrm{Av}(123, 231, 312)$.

Furthermore, $\mathcal{A}_0$ is ω-categorical and the model companion of $\mathrm{Th}(\mathrm{Av}(123, 231, 312))$.

Proof. The proof will be split up into three parts:

1. $\mathcal{A}$ is homogeneous.
2. $\mathrm{Age}(\mathcal{A}_0) \cong \mathrm{Av}(123, 231, 312)$.
3. $\mathcal{A}_0$ is ω-categorical, and its complete first-order theory is the model-companion of $\mathrm{Th}(\mathrm{Av}(123, 231, 312))$.

1. $\mathcal{A}$ is homogeneous. Let S, T be finite subsets of A, and $\mathcal{S}$, $\mathcal{T}$ substructures of $\mathcal{A}$ induced by these sets. Suppose that there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{T}$. In the following, it will be shown that ι can be extended to an automorphism on $\mathcal{A}$.

Consider the restrictions of ι to P, and N,

$$\iota_P = \iota\big|_{P \cap S} \qquad\qquad \iota_N = \iota\big|_{N \cap S}$$

Since ι is an isomorphism, the images of ι_P, and ι_N are solely in P, and N, respectively. Because of $(\mathbb{Q}, <)$ being a homogeneous structure, it is possible to extend ι_P, and ι_N to automorphisms η_P, and η_N on P, and N, respectively. Then the homomorphism $\eta : \mathcal{A} \to \mathcal{A}$ defined by

$$\eta(x) = \begin{cases} \eta_P(x), & \text{if } x \in P \\ \eta_N(x), & \text{if } x \in N \end{cases}$$

is an automorphism that extends ι.

Therefore, any isomorphism between finite substructures of $\mathcal{A}$ can be extended to an automorphism on $\mathcal{A}$. So $\mathcal{A}$ is homogeneous.

2. $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 231, 312)$. This part really splits up into two smaller statements. Namely, it is necessary to show that every structure in $\mathrm{Age}\,(\mathcal{A}_0)$ avoids the patterns 123, 231, and 312, and that for every permutation in $\mathrm{Av}(123, 231, 312)$ there exists a structure in $\mathrm{Age}\,(\mathcal{A}_0)$ isomorphic to it.

For the first part it helps to look more closely at the descents and ascents in the structures in $\mathrm{Age}\,(\mathcal{A}_0)$. Let $\mathcal{X} \in \mathrm{Age}\,(\mathcal{A}_0)$. Consider two elements $x, y \in \mathrm{dom}\,(\mathcal{X})$ of this structure with $x <_1 y$ and $x <_2 y$. From the former it follows that $x < y$. In combination with the latter this means that $x < 0$ and $0 < y$, and therefore, $x \in N$ and $y \in P$.

Similarly, consider two elements $x, y \in \mathrm{dom}\,(\mathcal{X})$ such that $x <_1 y$ and $y <_2 x$. Again, it follows that $x < y$. Together with the latter premise this implies that either $x, y \in P$, or $x, y \in N$.

Now suppose that $\mathcal{X}$ contains the pattern 123, that is, there exist $a, b, c \in \mathrm{dom}\,(\mathcal{X})$ with $a <_1 b <_1 c$ and $a <_2 b <_2 c$. Then it follows from the first observation that on one hand $b \in P$, but on the other, $b \in N$. This is obviously a contradiction.

Next, suppose that $\mathcal{X}$ contains the pattern 231, that is, there exist $a, b, c \in \mathrm{dom}\,(\mathcal{X})$ with $a <_1 b <_1 c$ and $b <_2 c <_2 a$. Then $b \in N$ and $c \in P$. This contradicts the conclusion from the second observation that either $a \in P$, and hence, $b, c \in P$, or $a \in N$, and therefore, $b.c \in N$.

Finally, suppose that $\mathcal{X}$ contains the pattern 312, that is, there exist $a, b, c \in \mathrm{dom}\,(\mathcal{X})$ with $a <_1 b <_1 c$ and $c <_2 a <_2 b$. Then the second observation implies that either $a, b, c \in P$ or $a, b, c \in N$. On the other hand, it follows from the first that $a \in N$ while $b \in P$. This is clearly a contradiction.

For the second part, consider a permutation $\pi \in \mathrm{Av}(123, 231, 312)$ of length n, which can be written in the form $\pi = p_1 p_2 \cdots p_n$. By Lemma 3.23 it is known that π is of the form

$$\pi = id^r_{n_1} \oplus id^r_{n_2}$$

for some $n_1, n_2 \in \mathbb{N}_0$ with $n_1 + n_2 = n$. Consider a substructure $\mathcal{B}$ of $\mathcal{A}$ generated by a finite subset B of A such that

$$|B \cap P| = n_2,$$
$$|B \cap N| = n_1.$$

Then B has n elements. Then the $\{<_1, <_2\}$-reduct $\mathcal{B}_0$ of $\mathcal{B}$ is isomorphic to π. This holds since on one hand

$$\forall\, x, y \in (B \cap P) : (x <_1 y) \to (y <_2 x),$$
$$\forall\, x, y \in (B \cap N) : (x <_1 y) \to (y <_2 x),$$
$$\forall\, x \in (B \cap P), y \in (B \cap N) : y <_1 x \wedge y <_2 x,$$

and on the other

$$\forall\, x, y \in \{n_1 + 1, \dots, n\} : (x <_1 y) \to (y <_2 x),$$
$$\forall\, x, y \in \{1, \dots, n_1\} : (x <_1 y) \to (y <_2 x),$$
$$\forall\, x, \in \{1, \dots, n_1\}, y \in \{n_1 + 1, \dots, n\} : x <_1 y \wedge x <_2 y.$$

Combining both parts yields that indeed $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 231, 312)$.

3. $\mathcal{A}_0$ is ω-categorical, and $\mathrm{Th}\,(\mathcal{A}_0)$ *is the model-companion of* $\mathrm{Th}\,(\mathrm{Av}(123, 231, 312))$. Since $\mathcal{A}$ is homogeneous, it is ω-categorical by Lemma 2.35. As $\mathcal{A}_0$ is its $\{<_1, <_2\}$-reduct, $\mathcal{A}_0$ is ω-categorical as well by Lemma 2.36.

To prove that $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete it is sufficient by Theorem 2.39 to show that $\mathcal{A}_0$ has a homogeneous expansion by countably many existentially definable relations whose complements are existentially definable in $\mathcal{A}_0$ as well.

Since $\mathcal{A}$ is homogeneous, and $\mathcal{A}_0$ is the $\{<_1, <_2\}$-reduct of $\mathcal{A}$, $\mathcal{A}_0$ does obviously have a homogeneous expansion by relations P, and N. It is easy to verify, that for these relations it is:

$$x \in P \iff \exists\, y : y <_1 x \wedge y <_2 x,$$
$$x \in N \iff \exists\, y : x <_1 y \wedge x <_2 y.$$

Note, that $N = A \setminus P$. Therefore, the relations themselves and their complements are existentially definable in $\mathcal{A}_0$. Hence, $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete. Since, $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 231, 312)$, this implies that $\mathrm{Th}\,(\mathcal{A}_0)$ is the model companion of $\mathrm{Th}\,(\mathrm{Av}(123, 231, 312))$. □

3.4.5 Class Av(132, 213, 231)

Before considering model theoretic properties, take a look at the following structural property of permutations in this class.

▻ **Lemma 3.26**
Let $\pi \in \mathrm{Av}(132, 213, 231)$ be an n-permutation. Then π is of the form

$$\pi = id^r_{n_1} \ominus id_{n_2}$$

for some $n_1, n_2 \in \mathbb{N}_0$, $n_1 + n_2 = n$.

Proof. Obviously, the identity permutation of length 1 is of the form described a both for $n_1 = 1$, $n_2 = 0$. Suppose that there exist some permutations in $\mathrm{Av}(132, 213, 231)$ that are not of the stated form. As the permutations all have finite lengths, there exists a not necessarily unique shortest permutation that is not of the stated form. Such permutations are at least of length 2.

Let $\sigma = s_1 \cdots s_m$ be a shortest permutation that is not of the form given above. Let i be the index of the largest entry of σ, that is, $s_i = m$. Then the permutation $\tau = s_1 \cdots s_{i-1}s_{i+1} \cdots s_m$ has length $m - 1 > 0$, and can be written as

$$\tau = id^r_{m_1} \ominus id_{m_2}$$

for some $m_1, m_2 \in \mathbb{N}_0$, $m_1 + m_2 = m - 1$. That is, $\tau = t_1, \ldots, t_{m-1}$ with $t_{m_1+1} < \ldots < t_{m-1} < t_{m_1} < \ldots < t_1$. Depending on the value of i the entry s_i would now be inserted into τ in different places when reconstructing σ from τ.

- If $i = 1$, then

$$\begin{aligned}\sigma &= (m, t_1, \ldots, t_{m-1}) = 1 \ominus \tau \\ &= 1 \ominus (id^r_{m_1} \ominus id_{m_2}) = (id^r_1 \ominus id^r_{m_1}) \ominus id_{m_2} \\ &= id^r_{m_1+1} \ominus id_{m_2}.\end{aligned}$$

  This contradicts the assumption.

- If $1 < i \leq m_1 + 1 < m$, then $\sigma = (t_1, \ldots, t_{i-1}, m, t_i, \ldots t_{m_1+1}, \ldots t_{m-1})$ with $t_i < t_{i-1} < m$. Then σ contains the pattern 231.

- If $m_1 + 1 < i < m$, then $\sigma = (t_1, \ldots, t_{m_i}, t_{m_1+1}, \ldots, t_{i-1}, m, t_i, \ldots, t_{m-1})$ with $t_{m_1+1} < t_i < m$. Then σ contains the pattern 132.

- If $i = m$, then $\sigma = (t_1, \ldots, t_{m-1}, m)$ with either $t_1 \leq t_{m-1} < m$, or $t_{m-1} < t_1 < m$. In the first case, σ would be isomorphic to the identity permutation of length m and therefore of the desired form. This contradicts the assumption. In the second case, σ would contain the pattern 213.

One way or another, all cases lead to a contradiction. Therefore, the assumption cannot hold. Hence, all permutations in the permutation pattern avoidance class Av(132, 213, 231) are of the form stated above. □

As for several of the previous classes it is possible to use this form of permutations in Av(132, 213, 231) to prove that the class has the joint embedding property. As the proof is fully analogous to previous proofs it will be omitted here.

▻ **Lemma 3.27**
Av(132, 213, 231) has the joint embedding property.

It is now possible to get a result with regard to amalgamation classes. The following theorem proves that Th (Av(132, 213, 231)) has an ω-categorical model companion.

▶ **Theorem 3.28**
Let $L = (<_1, <_2, P, N)$ be a relational signature where $<_1, <_2$ are binary, and P, N unary. Consider an L-structure $\mathcal{A}$ with domain

$$A = \mathbb{Q} \setminus \{0\},$$

and the relations defined such that

$$\begin{aligned} x \in P &\text{ if and only if } 0 < x, \\ x \in N &\text{ if and only if } x < 0, \\ x <_1 y &\text{ if and only if } x < y, \\ x <_2 y &\text{ if and only if } (0 < y \wedge y < x) \vee (y < 0 \wedge x < y) \vee (0 < x \wedge y < 0). \end{aligned}$$

Then $\mathcal{A}$ is homogeneous, and the age $\mathrm{Age}(\mathcal{A}_0)$ of its $\{<_1, <_2\}$-reduct $\mathcal{A}_0$ is isomorphic to Av(132, 213, 231).

Furthermore, $\mathcal{A}_0$ is ω-categorical and the model companion of Th (Av(132, 213, 231)).

Proof. The proof will be split up into three parts:

1. $\mathcal{A}$ is homogeneous.
2. $\mathrm{Age}(\mathcal{A}_0) \cong$ Av(132, 213, 231).
3. $\mathcal{A}_0$ is ω-categorical, and its complete first-order theory is the model-companion of Th (Av(132, 213, 231)).

1. $\mathcal{A}$ is homogeneous. Let S, T be finite subsets of A, and $\mathcal{S}$, $\mathcal{T}$ substructures of $\mathcal{A}$ induced by these sets. Suppose that there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{T}$. In the following, it will be shown that ι can be extended to an automorphism on $\mathcal{A}$.

Consider the restrictions of ι to P, and N,

$$\iota_P = \iota\big|_{P \cap S} \qquad \iota_N = \iota\big|_{N \cap S}$$

Since ι is an isomorphism, the images of ι_P, and ι_N are solely in P, and N, respectively. Because of $(\mathbb{Q}, <)$ being a homogeneous structure, it is possible to extend ι_P, and ι_N to automorphisms η_P, and η_N on P, and N, respectively. Then the homomorphism $\eta : \mathcal{A} \to \mathcal{A}$ defined by

$$\eta(x) = \begin{cases} \eta_P(x), & \text{if } x \in P \\ \eta_N(x), & \text{if } x \in N \end{cases}$$

is an automorphism that extends ι.

Therefore, any isomorphism between finite substructures of $\mathcal{A}$ can be extended to an automorphism on $\mathcal{A}$. So $\mathcal{A}$ is homogeneous.

2. $\operatorname{Age}(\mathcal{A}_0) \cong \operatorname{Av}(132, 213, 231)$. This part really splits up into two smaller statements. Namely, it is necessary to show that every structure in $\operatorname{Age}(\mathcal{A}_0)$ avoids the patterns 132, 213, and 231, and that for every permutation in $\operatorname{Av}(132, 213, 231)$ there exists a structure in $\operatorname{Age}(\mathcal{A}_0)$ isomorphic to it.

For the first part it helps to take a closer look at the descents and ascents in the structures in $\operatorname{Age}(\mathcal{A}_0)$. Let $\mathcal{X} \in \operatorname{Age}(\mathcal{A}_0)$. Consider two elements $x, y \in \operatorname{dom}(\mathcal{X})$ of this structure with $x <_1 y$ and $x <_2 y$. From the first premise it follows that $x < y$. In combination with the latter this means that $x < 0$ and $y < 0$. Therefore, $x, y \in N$.

Similarly, consider two elements $x, y \in \operatorname{dom}(\mathcal{X})$ such that $x <_1 y$ and $y <_2 x$. Again, it follows that $x < y$. Together with the second premise this implies that x is either positive or negative, and $0 < y$. Therefore, $y \in P$.

Suppose that $\mathcal{X}$ contains the pattern 132, that is, there exist $a, b, c \in \operatorname{dom}(\mathcal{X})$ with $a <_1 b <_1 c$ and $a <_2 c <_2 b$. Then it follows from the first observation that $a, b, c \in N$, and from the second that $c \in P$. This is obviously a contradiction.

Next, suppose that $\mathcal{X}$ contains the pattern 213, that is, there exist $a, b, c \in \operatorname{dom}(\mathcal{X})$ with $a <_1 b <_1 c$ and $b <_2 a <_2 c$. Then $a, b, c \in N$, and $b \in P$. Again, this is a contradiction.

Finally, suppose that $\mathcal{X}$ contains the pattern 231, that is, there exist $a, b, c \in \operatorname{dom}(\mathcal{X})$ with $a <_1 b <_1 c$ and $b <_2 c <_2 a$. Due to the first observation it is $b, c \in N$. On the other hand, the second implies $b, c \in P$. So there is a contradiction.

For the second part, consider a permutation $\pi \in \operatorname{Av}(132, 213, 231)$ of length n, which can be written as $\pi = p_1 p_2 \cdots p_n$. By Lemma 3.23 it is known that π is of the form

$$\pi = id_{n_1}^r \oplus id_{n_2}$$

for some $n_1, n_2 \in \mathbb{N}_0$ with $n_1 + n_2 = n$. Consider a substructure $\mathcal{B}$ of $\mathcal{A}$ generated by a finite subset B of A such that

$$|B \cap P| = n_1,$$
$$|B \cap N| = n_2.$$

Then B has n elements. Then the $\{<_1, <_2\}$-reduct $\mathcal{B}_0$ of $\mathcal{B}$ is isomorphic to π. This holds since on one hand

$$\forall x, y \in (B \cap P) : (x <_1 y) \to (y <_2 x),$$
$$\forall x, y \in (B \cap N) : (x <_1 y) \to (x <_2 y),$$
$$\forall x \in (B \cap P), y \in (B \cap N) : y <_1 x \wedge x <_2 y,$$

and on the other

$$\forall x, y \in \{n_2 + 1, \dots, n\} : (x <_1 y) \to (y <_2 x),$$
$$\forall x, y \in \{1, \dots, n_2\} : (x <_1 y) \to (x <_2 y),$$
$$\forall x, \in \{1, \dots, n_2\}, y \in \{n_2 + 1, \dots, n\} : x <_1 y \wedge y <_2 x.$$

Combining both parts yields that indeed $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(132, 213, 231)$.

3. $\mathcal{A}_0$ is ω-categorical, and $\mathrm{Th}\,(\mathcal{A}_0)$ *is the model-companion of* $\mathrm{Th}\,(\mathrm{Av}(132, 213, 231))$. Since $\mathcal{A}$ is homogeneous, it is ω-categorical by Lemma 2.35. As $\mathcal{A}_0$ is its $\{<_1, <_2\}$-reduct, $\mathcal{A}_0$ is ω-categorical as well by Lemma 2.36.

To prove that $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete it is sufficient by Theorem 2.39 to show that $\mathcal{A}_0$ has a homogeneous expansion by countably many existentially definable relations whose complements are existentially definable in $\mathcal{A}_0$ as well.

Since $\mathcal{A}$ is homogeneous, and $\mathcal{A}_0$ is the $\{<_1, <_2\}$-reduct of $\mathcal{A}$, $\mathcal{A}_0$ does obviously have a homogeneous expansion by relations P, and N. It is easy to verify, that for these relations it is:

$$x \in P \iff \exists\, y : y <_1 x \wedge x <_2 y,$$
$$x \in N \iff \exists\, y : y <_1 x \wedge y <_2 x.$$

Note, that $N = A \setminus P$. Therefore, the relations themselves and their complements are existentially definable in $\mathcal{A}_0$. Hence, $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete. Since, $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(132, 213, 231)$, this implies that $\mathrm{Th}\,(\mathcal{A}_0)$ is the model companion of $\mathrm{Th}\,(\mathrm{Av}(132, 213, 231))$. $\square$

3.5 Avoiding four patterns

In the previous two sections there were 5 different classes with respect to symmetry. Likewise, there are 5 classes here. According to [22] they are the ones that avoid the following sets of patterns.

- $\{123, 132, 213, 231\}$, $\{213, 231, 312, 321\}$, $\{123, 132, 213, 312\}$, $\{132, 231, 312, 321\}$
- $\{123, 132, 213, 321\}$, $\{123, 231, 312, 321\}$
- $\{123, 132, 231, 312\}$, $\{132, 213, 231, 321\}$, $\{123, 213, 231, 312\}$, $\{132, 213, 312, 321\}$

- $\{123, 132, 231, 321\}$, $\{123, 213, 231, 321\}$, $\{123, 213, 312, 321\}$, $\{123, 132, 312, 321\}$
- $\{132, 213, 231, 312\}$.

In this case there are two finite classes. All non-finite classes are Wilf-equivalent since they all have exactly two permutations of length n, $n > 1$. Now take a look at the model theoretic properties of the classes represented by the sets of patterns $\{123, 132, 213, 231\}$, $\{123, 132, 213, 321\}$, $\{123, 132, 231, 312\}$, $\{123, 132, 231, 321\}$ and $\{132, 213, 231, 312\}$.

3.5.1 Class Av(123, 132, 213, 231)

The following lemma gives very useful information about the form of the permutations in the permutation pattern avoidance class Av(123, 132, 213, 231).

▷ **Lemma 3.29**
Let $\pi \in \text{Av}(123, 132, 213, 231)$ be an n-permutation. Then π is either the reversal of the identity permutation of length n, that is,

$$\pi = (n, \dots, 1) = id_n^r,$$

or π is of the form

$$\pi = (n, \dots, 3, 1, 2) = id_{n-2}^r \ominus 12.$$

Proof. Let $\pi \in \text{Av}(123, 132, 213, 231)$. Since $\{123, 132, 231\} \subseteq \{123, 132, 213, 231\}$ it is $\text{Av}(123, 132, 213, 231) \subseteq \text{Av}(123, 132, 231)$. Therefore, by Lemma 3.20, π is of the form

$$\pi = id_{n_1}^r \ominus (id_{n_2}^r \oplus 1)$$

for some $n_1, n_2 \in \mathbb{N}_0$, $n_1 + n_2 + 1 = n$. If n_2 is equal to 0 or 1, this means that π has the desired form.

Now suppose that $n_2 > 1$. Then π has the substructure $21 \oplus 1 = 213$. That is, π contains the pattern 213. This contradicts the premise that π avoids said pattern. Hence, π has the form stated above. □

It is now very easy to proof that the joint embedding property holds.

▷ **Lemma 3.30**
Av(123, 132, 213, 231) has the joint embedding property.

Proof. Let $\pi, \phi \in \text{Av}(123, 132, 213, 231)$. Then by Lemma 3.29 the permutation

$$\psi = id_{|\pi|+|\pi|}^r \ominus 12$$

embeds both π and ϕ. Due to its structure ψ avoids all four patterns.

Hence, $\psi \in \text{Av}(123, 132, 213, 231)$. Therefore, the class has the joint embedding property. □

Now the following theorem proves that $\mathrm{Th}\,(\mathrm{Av}(123, 132, 213, 231))$ has an ω-categorical model companion. The approach is the same as for previous examples.

Theorem 3.31
Let $L = (<_1, <_2, P, E, Z)$ be a relational signature where $<_1, <_2$ are binary, and P, E, Z unary. Let $\mathcal{A}$ be an L-structure with domain

$$A = \mathbb{Q}^+ \cup \{-1, 0\},$$

and the relations defined such that

$$\begin{aligned} x \in P &\text{ if and only if } 0 < x, \\ x \in E &\text{ if and only if } x = -1, \\ x \in Z &\text{ if and only if } x = 0, \\ x <_1 y &\text{ if and only if } x < y, \\ x <_2 y &\text{ if and only if } (0 < x \wedge y < x) \vee (x = 0 \wedge y = -1). \end{aligned}$$

Then $\mathcal{A}$ is homogeneous, and the age $\mathrm{Age}\,(\mathcal{A}_0)$ of its $\{<_1, <_2\}$-reduct $\mathcal{A}_0$ is isomorphic to $\mathrm{Av}(123, 132, 213, 231)$. Furthermore, $\mathcal{A}_0$ is ω-categorical and the model companion of $\mathrm{Th}\,(\mathrm{Av}(123, 132, 213, 231))$.

Proof. First, note that P, E, and Z form a partition of A. This will be important throughout the proof, but will not always be explicitly stated.

The proof will be split up into three parts:

1. $\mathcal{A}$ is homogeneous.
2. $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 213, 231)$.
3. $\mathcal{A}_0$ is ω-categorical, and its complete first-order theory is the model-companion of $\mathrm{Th}\,(\mathrm{Av}(123, 132, 213, 231))$.

1. $\mathcal{A}$ is homogeneous. Let S, T be finite subsets of A, and $\mathcal{S}$, $\mathcal{T}$ substructures of $\mathcal{A}$ induced by these sets. Suppose that there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{T}$. It needs to be proven now that ι can be extended to an automorphism on $\mathcal{A}$.

Consider the restriction of ι to P,

$$\iota_P = \iota\big|_{P \cap S}$$

Since ι is an isomorphism, the image of ι_P is in P. Note also, that due to the definition of $\mathcal{A}$ if $0 \in S$, $\iota(0) = 0$, and analogously, if $-1 \in S$, then $\iota(-1) = -1$.

Because of $(\mathbb{Q}, <)$ being a homogeneous structure, it is possible to extend ι_P to an automorphism η_P on P. Then the homomorphism $\eta : \mathcal{A} \to \mathcal{A}$ defined by

$$\eta(x) = \begin{cases} \eta_P(x), & \text{if } x \in P \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x = -1 \end{cases}$$

is an automorphism that extends ι.

Therefore, any isomorphism between finite substructures of $\mathcal{A}$ can be extended to an automorphism on $\mathcal{A}$. So $\mathcal{A}$ is homogeneous.

2. $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 213, 231)$. This part splits up into two smaller statements. Namely, it must be shown that every structure in $\mathrm{Age}\,(\mathcal{A}_0)$ avoids the patterns 123, 132, 213 and 231, and that for every permutation in $\mathrm{Av}(123, 132, 213, 231)$ there exists a structure in $\mathrm{Age}\,(\mathcal{A}_0)$ isomorphic to it.

For the first part it helps to look more closely at the descents and ascents in the structures in $\mathrm{Age}\,(\mathcal{A}_0)$. Let $\mathcal{X} \in \mathrm{Age}\,(\mathcal{A}_0)$. Consider two elements $x, y \in \mathrm{dom}\,(\mathcal{X})$ of this structure with $x <_1 y$ and $x <_2 y$. From the former it follows that $x < y$. In combination with the latter this means that $x = -1$ and $y = 0$. Therefore, it has to be $x \in E$ and $y \in Z$.

Similarly, consider two elements $x, y \in \mathrm{dom}\,(\mathcal{X})$ such that $x <_1 y$ and $y <_2 x$. Again, it follows from that $x < y$. Together with the second premise this implies that $0 < y$. Therefore, $y \in P$.

Suppose that $\mathcal{X}$ contains the pattern 123, that is, there are $r, s, t \in \mathrm{dom}\,(\mathcal{X})$ with $r <_1 s <_1 t$ and $r <_2 s <_2 t$. Since $r <_1 s$ and $r <_2 s$ as well as $s <_1 t$ and $s <_2 t$, it has to be $s \in Z$, and simultaneously $s \in E$. But this contradicts the fact that by definition E and Z are disjoint.

Now suppose that $\mathcal{X}$ contains the pattern 132, that is, there exist $r, s, t \in \mathrm{dom}\,(\mathcal{X})$ with $r <_1 s <_1 t$ and $r <_2 t <_2 s$. From the first observation it follows that $s, t \in Z$. Hence, $s = t = 0$. This contradicts the assumption that s and t are distinct.

Next, suppose that $\mathcal{X}$ contains the pattern 213, that is, there exist $r, s, t \in \mathrm{dom}\,(\mathcal{X})$ with $r <_1 s <_1 t$ and $s <_2 r <_2 t$. It follows from the first observation that $r, s \in E$, that is, $r = s = -1$. But this is a contradiction.

Finally, suppose that $\mathcal{X}$ contains the pattern 231, that is, there exist $r, s, t \in \mathrm{dom}\,(\mathcal{X})$ with $r <_1 s <_1 t$ and $s <_2 t <_2 r$. The second observation implies that $s, t \in P$. On the other hand, it follows from the first that $s \in E$ and $t \in Z$. Once again, this yields a contradiction.

For the second part, consider a permutation $\pi \in \mathrm{Av}(123, 132, 213, 231)$ of length n, $\pi = p_1 p_2 \cdots p_n$. If the entries of π are strictly decreasing, it is straightforward that π is of the form $(n, \ldots, 1)$ and isomorphic to an arbitrary substructure of $A \cap P$ with n elements. By Lemma 3.29 it is known that the only other possible case is that π is of the form $(n, \ldots, n-2, 1, 2)$. Then by the definition of $\mathcal{A}$ any substructure whose domain contains -1, 0 and $n-2$ elements that are in $A \cap P$ is isomorphic to π.

Combining both statements yields that indeed $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 213, 231)$.

3. $\mathcal{A}_0$ *is* ω*-categorical, and* $\mathrm{Th}\,(\mathcal{A}_0)$ *is the model-companion of* $\mathrm{Th}\,(\mathrm{Av}(123, 132, 213, 231))$.

Since $\mathcal{A}$ is homogeneous, it is ω-categorical by Lemma 2.35. As $\mathcal{A}_0$ is its $\{<_1, <_2\}$-reduct, $\mathcal{A}_0$ is ω-categorical as well by Lemma 2.36.

To prove that $\mathrm{Th}(\mathcal{A}_0)$ is model-complete it is sufficient by Theorem 2.39 to show that $\mathcal{A}_0$ has a homogeneous expansion by countably many existentially definable relations whose complements are existentially definable in $\mathcal{A}_0$ as well.

Since $\mathcal{A}$ is homogeneous, and $\mathcal{A}_0$ is the $\{<_1, <_2\}$-reduct of $\mathcal{A}$, $\mathcal{A}_0$ does obviously have a homogeneous expansion by relations P, E, and Z. It is easy to verify that for these relations it is:

$$\begin{aligned} x \in P &\iff \exists\, y : y <_1 x \wedge x <_2 y \\ x \in A \setminus P &\iff \exists\, y : (x <_1 y \wedge x <_2 y) \vee (y <_1 x \wedge y <_2 x) \end{aligned}$$

$$\begin{aligned} x \in E &\iff \exists\, y : x <_1 y \wedge x <_2 y \\ x \in A \setminus E &\iff \exists\, y : (y <_1 x \wedge x <_2 y) \vee (y <_1 x \wedge y <_2 x) \end{aligned}$$

$$\begin{aligned} x \in Z &\iff \exists\, y : y <_1 x \wedge y <_2 x \\ x \in A \setminus Z &\iff \exists\, y, z : (y <_1 x \wedge x <_2 y) \vee (x <_1 y \wedge x <_2 y) \end{aligned}$$

Therefore, the relations themselves and their complements are existentially definable in $\mathcal{A}_0$. Hence, $\mathrm{Th}(\mathcal{A}_0)$ is model-complete. Since, $\mathrm{Age}(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 213, 231)$, this implies that $\mathrm{Th}(\mathcal{A}_0)$ is the model companion of $\mathrm{Th}(\mathrm{Av}(123, 132, 213, 231))$. □

3.5.2 Class Av(123, 132, 213, 321)

By Corollary 3.13 this permutation pattern avoidance class is finite. Closer inspection yields

$$\mathrm{Av}(123, 132, 213, 321) = \{1, 12, 21, 231, 312, 3412\}.$$

This class does have the joint embedding property. More specifically, all permutations in this class are substructures of the permutation 3412. Since the class is finite it will not be considered in more detail here.

3.5.3 Class Av(123, 132, 231, 312)

The following lemma shows that permutations in this class also have a very specific inner structure.

▻ Lemma 3.32

Let $\pi \in \mathrm{Av}(123, 132, 231, 312)$ be an n-permutation. Then π is either the reversal of the identity permutation of length n, that is,

$$\pi = (n, \ldots, 1) = id_n^r,$$

or π is of the form

$$\pi = (n-1, \ldots, 1, n) = id_{n-1}^r \oplus 1.$$

Proof. Let $\pi \in \mathrm{Av}(123, 132, 231, 312)$. Since $\{123, 132, 231\} \subseteq \{123, 132, 213, 231\}$ it is $\mathrm{Av}(123, 132, 231, 312) \subseteq \mathrm{Av}(123, 132, 231)$. Therefore, by Lemma 3.20, π is of the form

$$\pi = id_{n_1}^r \ominus (id_{n_2}^r \oplus 1)$$

for some $n_1, n_2 \in \mathbb{N}_0$, $n_1 + n_2 + 1 = n$. If $n_1 = 0$, or $n_2 = 0$, then π has the desired form.

Now suppose that $n_1, n_2 > 0$. Then π has the substructure $1 \ominus (1 \oplus 1) = 312$. That is, π contains the pattern 312. This contradicts the premise that π avoids said pattern. Hence, π has the form stated above. □

As in previous examples, this structure guarantees that $\mathrm{Av}(123, 132, 231, 312)$ has the joint embedding property. Now the following theorem states that, moreover, its first-order theory has an ω-categorical model companion.

► Theorem 3.33
Let $L = (<_1, <_2, N, Z)$ be a relational signature where $<_1, <_2$ are binary, and N, Z unary. Let $\mathcal{A}$ be an L-structure with domain

$$A = \mathbb{Q} \setminus \mathbb{Q}^+,$$

and the relations defined such that

$$\begin{aligned} x \in N &\text{ if and only if } x < 0, \\ x \in Z &\text{ if and only if } x = 0, \\ x <_1 y &\text{ if and only if } x < y, \\ x <_2 y &\text{ if and only if } (x < 0 \wedge y < x) \vee (x < 0 \wedge y = 0). \end{aligned}$$

Then $\mathcal{A}$ is homogeneous, and the age $\mathrm{Age}(\mathcal{A}_0)$ of its $\{<_1, <_2\}$-reduct $\mathcal{A}_0$ is isomorphic to $\mathrm{Av}(123, 132, 231, 312)$. Furthermore, $\mathcal{A}_0$ is ω-categorical and its complete first-order theory is the model companion of $\mathrm{Th}(\mathrm{Av}(123, 132, 231, 312))$.

Proof. First, note that P, E, and Z form a partition of A. This will be important throughout the proof, but will not always be explicitly stated.

The proof will be split up into three parts:

1. $\mathcal{A}$ is homogeneous.
2. $\mathrm{Age}(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 231, 312)$.
3. $\mathcal{A}_0$ is ω-categorical, and its complete first-order theory is the model-companion of $\mathrm{Th}(\mathrm{Av}(123, 132, 231, 312))$.

1. $\mathcal{A}$ is homogeneous. Let S, T be finite subsets of A, and $\mathcal{S}$, $\mathcal{T}$ the substructures of $\mathcal{A}$ induced by these sets. Suppose that there exists an isomorphism $\iota : \mathcal{S} \to \mathcal{T}$. Now one needs to show that ι can be extended to an automorphism on $\mathcal{A}$.

Consider the restriction of ι to N,

$$\iota_N = \iota\big|_{N\cap S}$$

Since ι is an isomorphism, the image of ι_N is in N. Note also, that due to the definition of $\mathcal{A}$ if $0 \in S$, $\iota(0) = 0$.

Because of $(\mathbb{Q}, <)$ being a homogeneous structure, it is possible to extend ι_N to an automorphism η_N on N. Then the homomorphism $\eta : \mathcal{A} \to \mathcal{A}$ defined by

$$\eta(x) = \begin{cases} \eta_N(x), & \text{if } x \in N \\ 0, & \text{if } x = 0 \end{cases}$$

is an automorphism that extends ι.

Therefore, any isomorphism between finite substructures of $\mathcal{A}$ can be extended to an automorphism on $\mathcal{A}$. So $\mathcal{A}$ is homogeneous.

2. $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 231, 312)$. This part splits up into two smaller statements. Namely, it must be shown that every structure in $\mathrm{Age}\,(\mathcal{A}_0)$ avoids the patterns 123, 132, 213 and 231, and that for every permutation in $\mathrm{Av}(123, 132, 231, 312)$ there exists a structure in $\mathrm{Age}\,(\mathcal{A}_0)$ isomorphic to it.

For the first part it helps to look more closely at the descents and ascents in the structures in $\mathrm{Age}\,(\mathcal{A}_0)$. Let $\mathcal{X} \in \mathrm{Age}\,(\mathcal{A}_0)$. Consider two elements $x, y \in \mathrm{dom}\,(\mathcal{X})$ of this structure with $x <_1 y$ and $x <_2 y$. From the former it follows that $x < y$. In combination with the latter this means that $x < 0$ and $y = 0$. Therefore, observe that in this case it has to be $x \in N$ and $y \in Z$.

Similarly, consider two elements $x, y \in \mathrm{dom}\,(\mathcal{X})$ such that $x <_1 y$ and $y <_2 x$. Again, it follows that $x < y$. This time the second premise implies that $x, y < 0$. Therefore, $x, y \in N$.

Now suppose that $\mathcal{X}$ contains the pattern 123, that is, there are $r, s, t \in \mathrm{dom}\,(\mathcal{X})$ with $r <_1 s <_1 t$ and $r <_2 s <_2 t$. Since $r <_1 s$ and $r <_2 s$ as well as $s <_1 t$ and $s <_2 t$, it has to be $s \in Z$, and simultaneously $s \in N$. But this contradicts the fact that by definition N and Z are disjoint.

Suppose that $\mathcal{X}$ contains the pattern 132, that is, there exist $r, s, t \in \mathrm{dom}\,(\mathcal{X})$ with $r <_1 s <_1 t$ and $r <_2 t <_2 s$. From the first observation it follows that $s, t \in Z$. Hence, $s = t = 0$. This contradicts the direct implication from the assumption that s and t are distinct.

Next, suppose that $\mathcal{X}$ contains the pattern 231, that is, there exist $r, s, t \in \mathrm{dom}\,(\mathcal{X})$ with $r <_1 s <_1 t$ and $s <_2 t <_2 r$. The second observation implies that $r, s, t \in N$, while it follows from the first that $t \in Z$. This is obviously a contradiction.

Finally, suppose that $\mathcal{X}$ contains the pattern 312, that is, there exist $r, s, t \in \mathrm{dom}\,(\mathcal{X})$ with

$r <_1 s <_1 t$ and $t <_2 r <_2 s$. Again, it follows from the second observation that $r, s, t \in N$, and from the first that $s \in Z$, which is a contradiction.

For the second part, consider a permutation $\pi \in \mathrm{Av}(123, 132, 231, 312)$ of length n, $\pi = p_1 p_2 \cdots p_n$. If the entries of π are strictly decreasing, it is straightforward that π is of the form $(n, \ldots, 1)$ and isomorphic to an arbitrary substructure of $A \cap N$ with n elements. By Lemma 3.32 it is known that the only other possible case is that π is of the form $(n-1, \ldots, 1, n)$. Then by the definition of $\mathcal{A}$ any substructure whose domain contains 0 and $n-1$ elements that are in $A \cap N$ is isomorphic to π.

Combining both statements yields that indeed $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 231, 312)$.

3. $\mathcal{A}_0$ is ω-categorical, and $\mathrm{Th}\,(\mathcal{A}_0)$ *is the model-companion of* $\mathrm{Th}\,(\mathrm{Av}(123, 132, 231, 312))$. Since $\mathcal{A}$ is homogeneous, it is ω-categorical by Lemma 2.35. As $\mathcal{A}_0$ is its $\{<_1, <_2\}$-reduct, $\mathcal{A}_0$ is ω-categorical as well by Lemma 2.36.

To prove that $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete it is sufficient by Theorem 2.39 to show that $\mathcal{A}_0$ has a homogeneous expansion by countably many existentially definable relations whose complements are existentially definable in $\mathcal{A}_0$ as well.

Since $\mathcal{A}$ is homogeneous, and $\mathcal{A}_0$ is the $\{<_1, <_2\}$-reduct of $\mathcal{A}$, $\mathcal{A}_0$ does obviously have a homogeneous expansion by relations N and Z. It is easy to verify that for these relations it is:

$$x \in N \iff \exists\, y : x <_1 y \wedge x <_2 y$$
$$x \in Z \iff \exists\, y : y <_1 x \wedge y <_1 x$$

Note also, that $Z = A \setminus N$. Therefore, the relations themselves and their complements are existentially definable in $\mathcal{A}_0$.

Hence, $\mathrm{Th}\,(\mathcal{A}_0)$ is model-complete. Since, $\mathrm{Age}\,(\mathcal{A}_0) \cong \mathrm{Av}(123, 132, 231, 312)$, this implies that $\mathrm{Th}\,(\mathcal{A}_0)$ is the model companion of $\mathrm{Th}\,(\mathrm{Av}(123, 132, 231, 312))$. □

3.5.4 Class Av(123, 132, 231, 321)

This permutation pattern avoidance class is finite by Corollary 3.13. It can be observed that

$$\mathrm{Av}(123, 132, 231, 321) = \{1, 12, 21, 213, 312\}.$$

Unlike the other finite class that avoids four patterns of length 3 this one does not have the joint embedding property. For example, there is no permutation that has both 213 and 312 as substructures.

3.5.5 Class Av(132, 213, 231, 312)

As this class avoids all patterns of length 3 except for the identity permutation and its reversal, its elements have one of two possible forms. They are either an identity permutation or a reversal of one.

Unfortunately, but rather obviously, this class does not have the joint embedding property. For example, there is no permutation in it that contains both 123 and 321 as a pattern.

3.6 Avoiding more than four patterns

The following are the two groups of symmetric sets of five patterns of length 3.

- $\{123, 132, 213, 231, 312\}$, $\{132, 213, 231, 312, 321\}$
- $\{123, 132, 213, 231, 321\}$, $\{123, 132, 213, 312, 321\}$, $\{123, 132, 231, 312, 321\}$, $\{123, 213, 231, 312, 321\}$

While the second defines a finite permutation pattern avoidance class, the classes associated with the first have exactly one permutation of each length except for $n = 2$ where there are two permutations of length 2.

Obviously, there is only one way to choose six patterns of length 3. That is the set

- $\{123, 132, 213, 231, 312, 321\}$

of all permutations of length 3. Obviously, this defines a finite permutation pattern avoidance class.

3.6.1 Class Av(123, 132, 213, 231, 312)

Unlike the other class forbidding five patterns of length 3, this class contains infinitely many elements. It consists of all permutations up to length 2, and all reversals of identity permutations of arbitrary length since 321 is the only pattern of length 3 that does not have to be avoided.

This class does not have the joint embedding property, though, as there is no structure which contains both 12 and 21 as patterns. Therefore, it will not be considered further.

3.6.2 Class Av(123, 132, 213, 231, 321)

Among others, this class avoids the patterns 123 and 321. Hence, it is finite by Corollary 3.13. Closer inspection reveals that it contains exactly the following permutations:

$$\mathrm{Av}(123, 132, 213, 231, 321) = \{1, 12, 21, 312\}$$

Surprisingly, this class actually has the joint embedding property since all elements can be embedded into the permutation 312. But as it is finite, it will not be considered further.

3.6.3 Class Av(123, 132, 213, 231, 312, 321)

As this class avoids all permutations of length 3 it is finite and contains precisely all permutations up to length 2. Therefore,

$$\text{Av}(123, 132, 213, 231, 312, 321) = \{1, 12, 21\}.$$

It is easy to see that the joint embedding property does not hold here as there is no permutation in this class into which 12 and 21 could be embedded.

4 Conclusions

While not the expected outcome, it was possible for me to proof for all but two of the considered classes that if they have the joint embedding property, but are not finite, then their first-order theory has an ω-categorical model companion. The same thing might also be true for the two classes for which I have not been able to find a proof yet.

This lead to the following conjecture.

▻ **Conjecture 4.1 – Bodirsky, 2015**
If a permutation pattern avoidance class has the joint embedding property and is not finite, then its complete first-order theory has an ω-categorical model companion.

Basically, there were two approaches used to prove this property in the specific cases treated in the previous chapter. In both cases structures with an expanded relational signature were considered. While in the first version the focus was on finding an appropriate subset of the rational numbers to model the permutation pattern avoidance class on, the second tried to find an amalgamation class that contained the respective permutation pattern class as a reduct while not explicitly defining the domain of the structures in the amalgamation class. Although the first approach was easier to get behind for specific examples, it seems to me that the second may prove to be easier to generalise as it depends far less on the class at hand.

If this conjecture should prove to be true, it would have a number of consequences. Among other things this would mean that if a permutation pattern avoidance class that avoids $\mathcal{F}$ has the joint embedding property and not just finitely many members, then there exists a universal limit that embeds not only all finite $\mathcal{F}$-avoiding structures, but also all countable $\mathcal{F}$-avoiding structures. While the existence of a structure that embeds all finite $\mathcal{F}$-avoiding structures directly follows from the joint embedding property by Lemma 2.27, this would mean that the joint embedding property is much stronger for permutation pattern avoidance classes.

Another problem that has not yet been solved is that of recognising permutation pattern avoidance classes that have the joint embedding property. Up to now no effective algorithm exists to verify whether a given permutation pattern avoidance class has the joint embedding property or not.

So while this thesis does not answer any general questions, it does show the implications that the joint embedding property has for some specific examples. It also strongly suggests that the conjecture stated above holds. Nonetheless, there are multiple ways in which this work could be carried on, some of which were discussed above.

Appendix A. Further Illustrations

The following illustrations were made while pursuing the topic, but did not make it into the main part of the thesis. Nonetheless, they greatly help imagination and are therefore presented here.

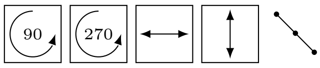
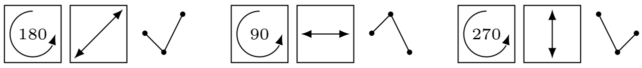
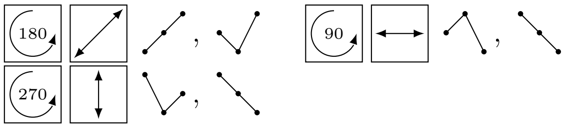
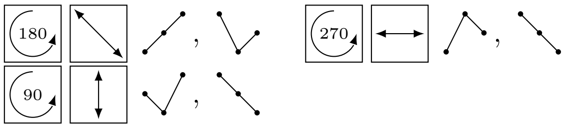
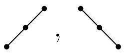
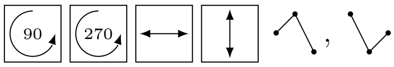
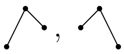
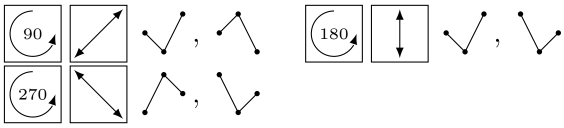
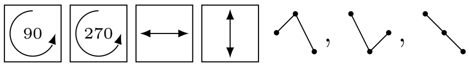
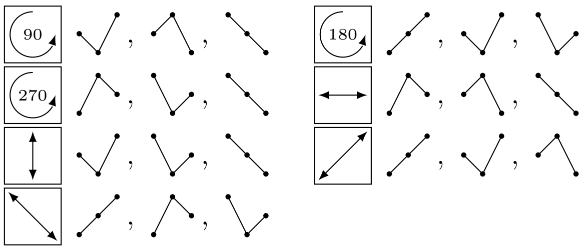
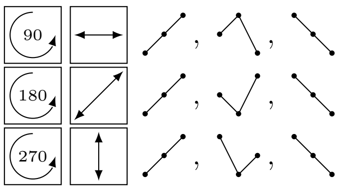
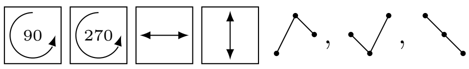
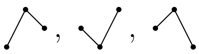
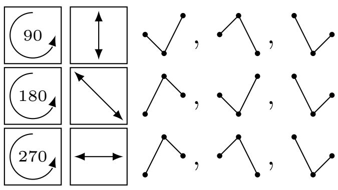
Table A.1 shows an illustrated version of the symmetry classes for sets of patterns of length 3 as presented at the beginning of each section in Chapter 3. The symbols next to the sets of permutations indicate the symmetry in D_4. It also introduces a numbering for the classes.

Table A.2 gives some examples for permutations in the respective avoidance class. It also contains some notes on the possible limit objects. Figure A.1 nicely illustrates the limit structures of those infinite permutation pattern avoidance classes avoiding patterns of length 3 that have the joint embedding property. The only class for which no nice representation of the limit was found is Class 1.2.

Class 1.2 has a tree-like structure, which can be seen from the observation that the minimal components in its $\ominus$-decomposition are either isomorphic to 1, or if they have length $n > 1$ then they are isomorphic to a permutation of the form $\phi \oplus 1$ with $\phi \in S_{n-1}$ such that the $\ominus$-decomposition of ϕ also has the stated property. Figure A.2 attempts to illustrate this.

Last but not least, Figure A.3 illustrates the inclusion relation between the symmetric sets of avoided patterns using a Hasse diagram. Here a set is included in another set if it is a subset of one of the sets in the respective symmetry class. Furthermore, all finite classes, all classes that have a Fraïssé limit, and the classes satisfying the joint embedding property are marked. If one class lies below another class in the diagram then this means that for any representatives A and B of the two symmetry classes $A \subset B$ it is $\mathrm{Av}(B) \subseteq \mathrm{Av}(A)$.

Table A.1. Sets of patterns of length 3; the first column gives a numbering, while the second column shows a representative for the respective symmetric sets and the third lists all other sets symmetric to the representative indicating the symmetry.

	set	equivalent sets
0.1	∅	none
1.1		
1.2		
2.1		
2.2		
2.3		none
2.4		
2.5		
3.1		
3.2		
3.3		
3.4		
3.5		

→ *continued on next page*

Table A.1 – continued from previous page

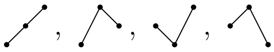
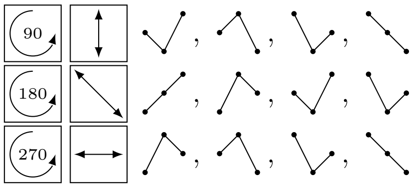
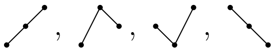
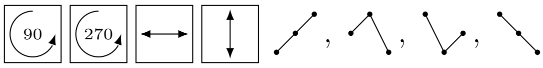
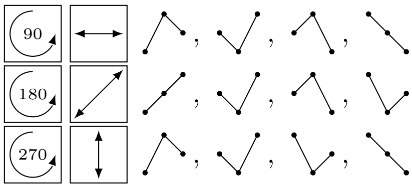
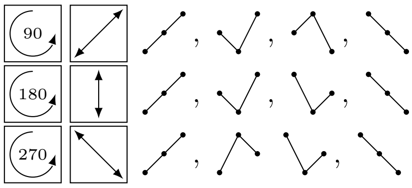
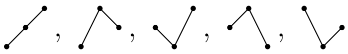
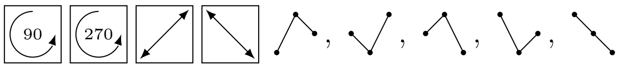
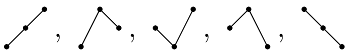
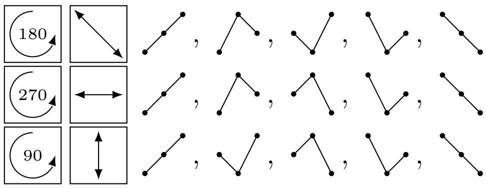
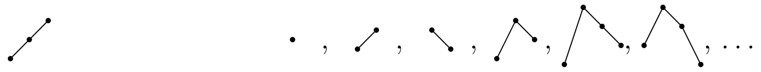
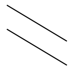
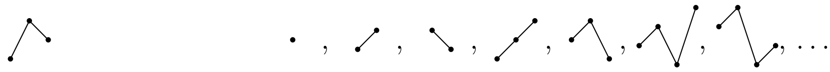
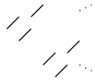
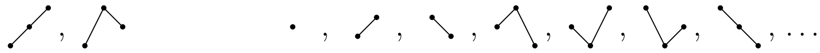
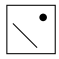
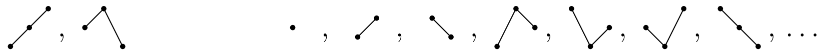
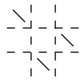
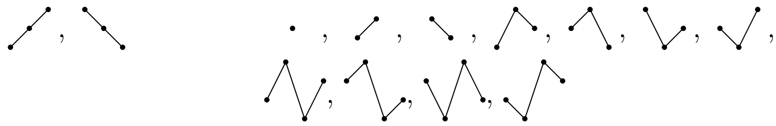
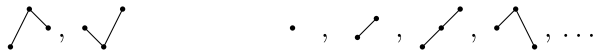
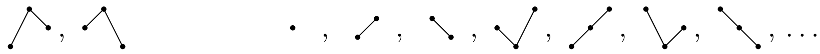
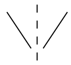
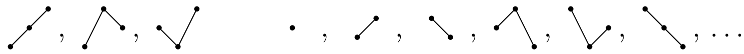
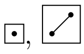
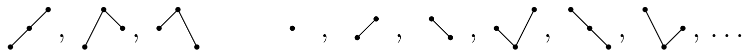
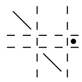
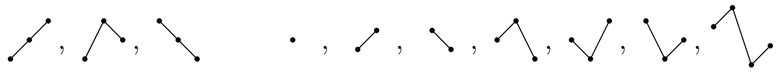
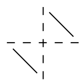
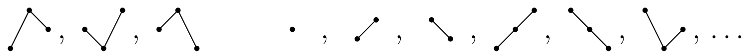
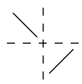
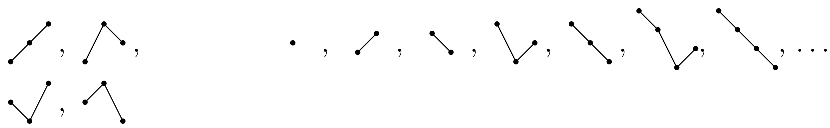
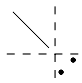
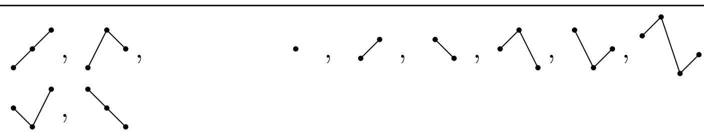
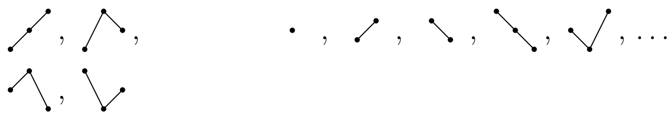
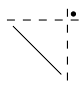
	set	equivalent sets
4.1		
4.2		
4.3		
4.4		
4.5		none
5.1		
5.2		
6.1		none

Table A.2. Some examples for elements of the permutation pattern avoidance class corresponding to the set of avoided patterns as stated as the representative for each symmetry class given in Table A.1 as well as notes on the limit object of each class.

0.1	∅ — class contains all permutations	Fraïssé class
1.1		
1.2		
2.1		⊖-sum of 
2.2		
2.3		finite
2.4		⊖-sum of , Fraïssé
2.5		
3.1		⊖-sum of 
3.2		
3.3		finite
3.4		
3.5		
4.1		
4.2		finite
4.3		

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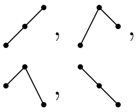
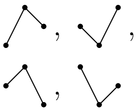
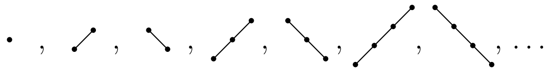
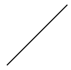
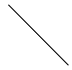
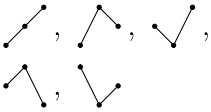
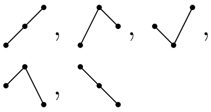
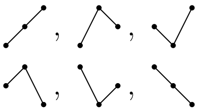
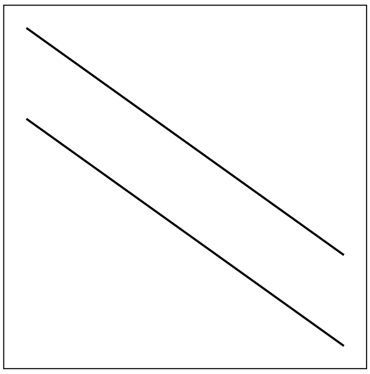
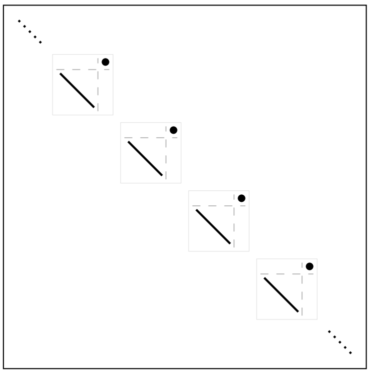
4.4			finite
4.5			 or 
5.1			 or 
5.2			finite
6.1			finite

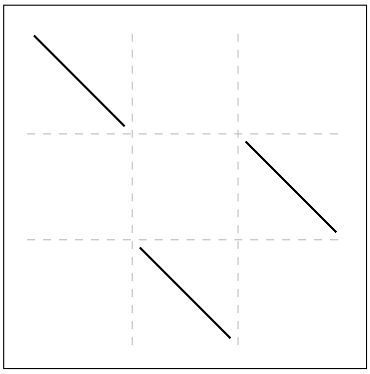


(a) Class 1.1
Av(123)

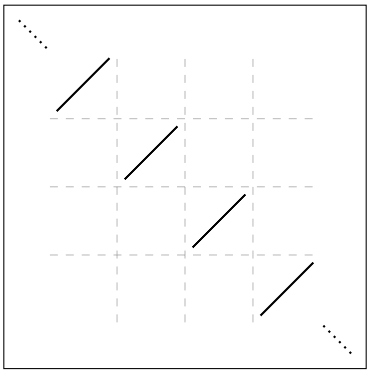
(b) Class 1.2
Av(132)



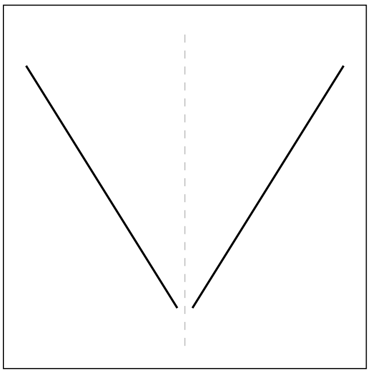
(c) Class 2.1
Av(123, 132)



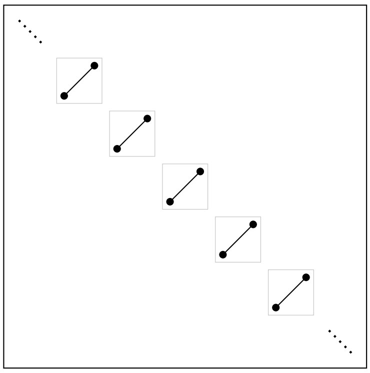
(d) Class 2.2
Av(123, 231)



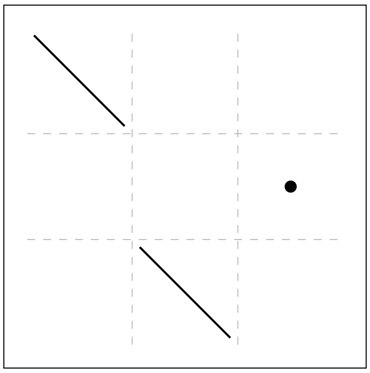
(e) Class 2.4 (Fraïssé)
Av(132, 213)



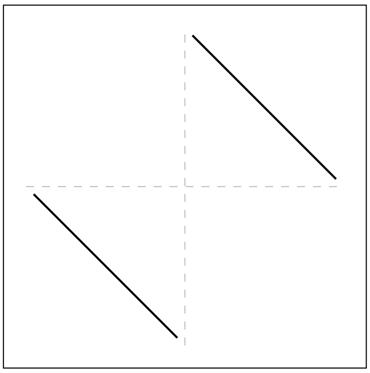
(f) Class 2.5
Av(132, 231)



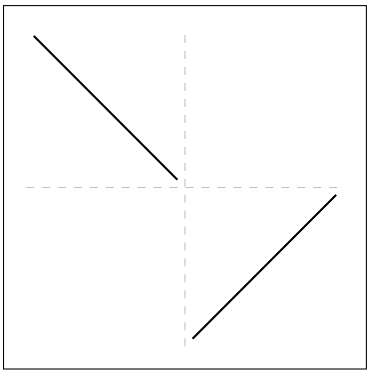
(g) Class 3.1
Av(123, 132, 213)



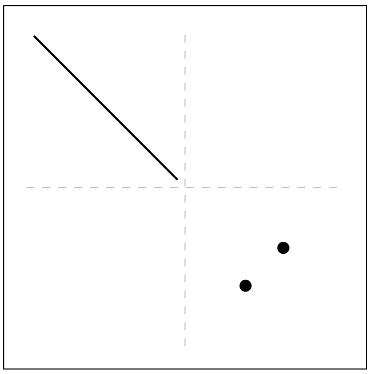
(h) Class 3.2
Av(123, 132, 231)



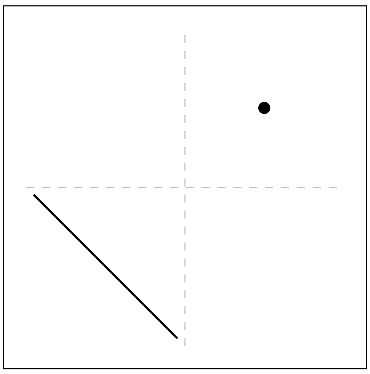
(i) Class 3.4
Av(123, 231, 312)



(j) Class 3.5
Av(132, 213, 231)



(k) Class 4.1
Av(123, 132, 213, 231)



(l) Class 4.3
Av(123, 132, 231, 312)

Figure A.1. Limit structures of infinite permutation pattern avoidance classes that have the joint embedding property.

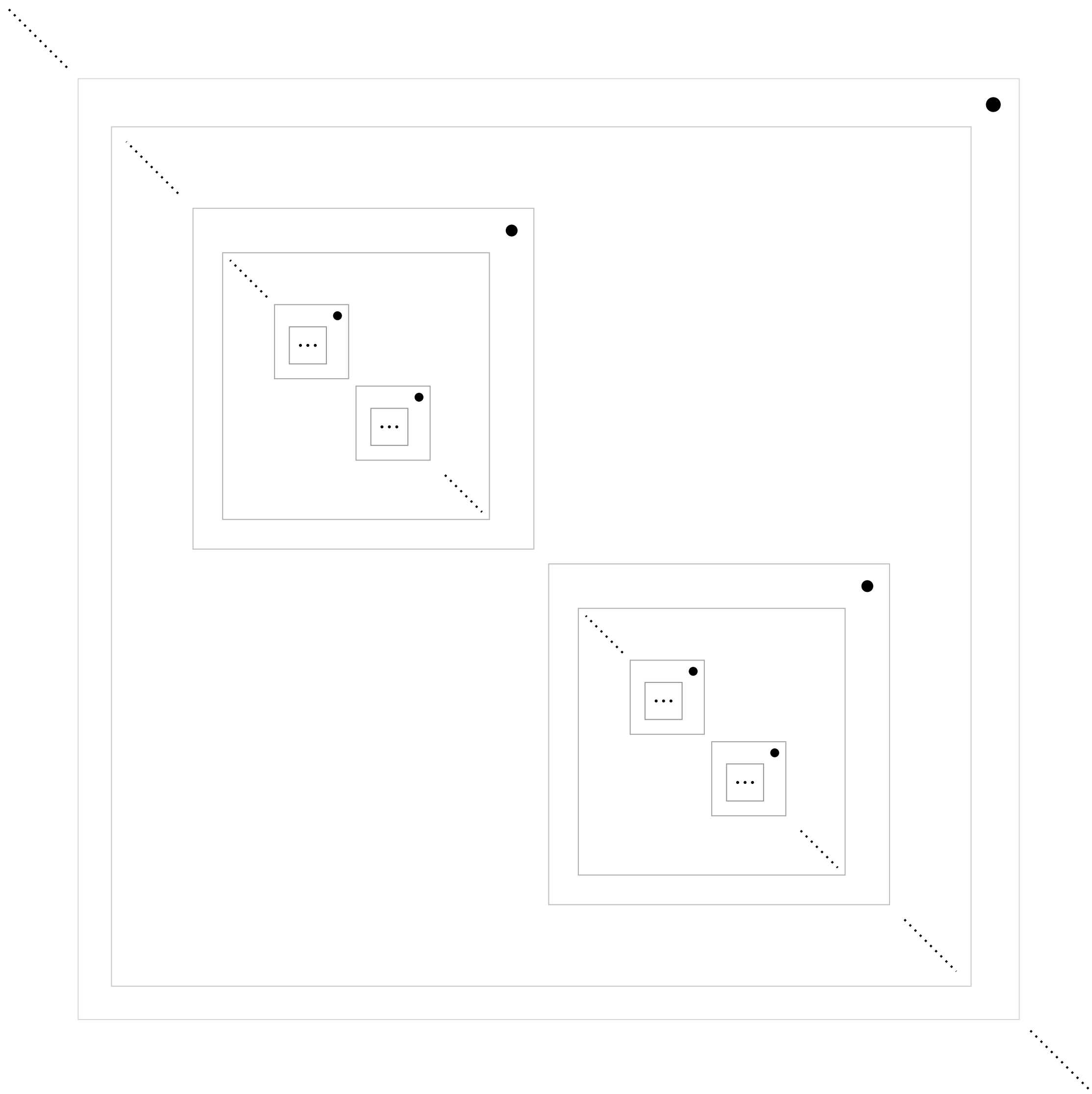


Figure A.2. Recursive structure of permutations in the avoidance class Av(132).

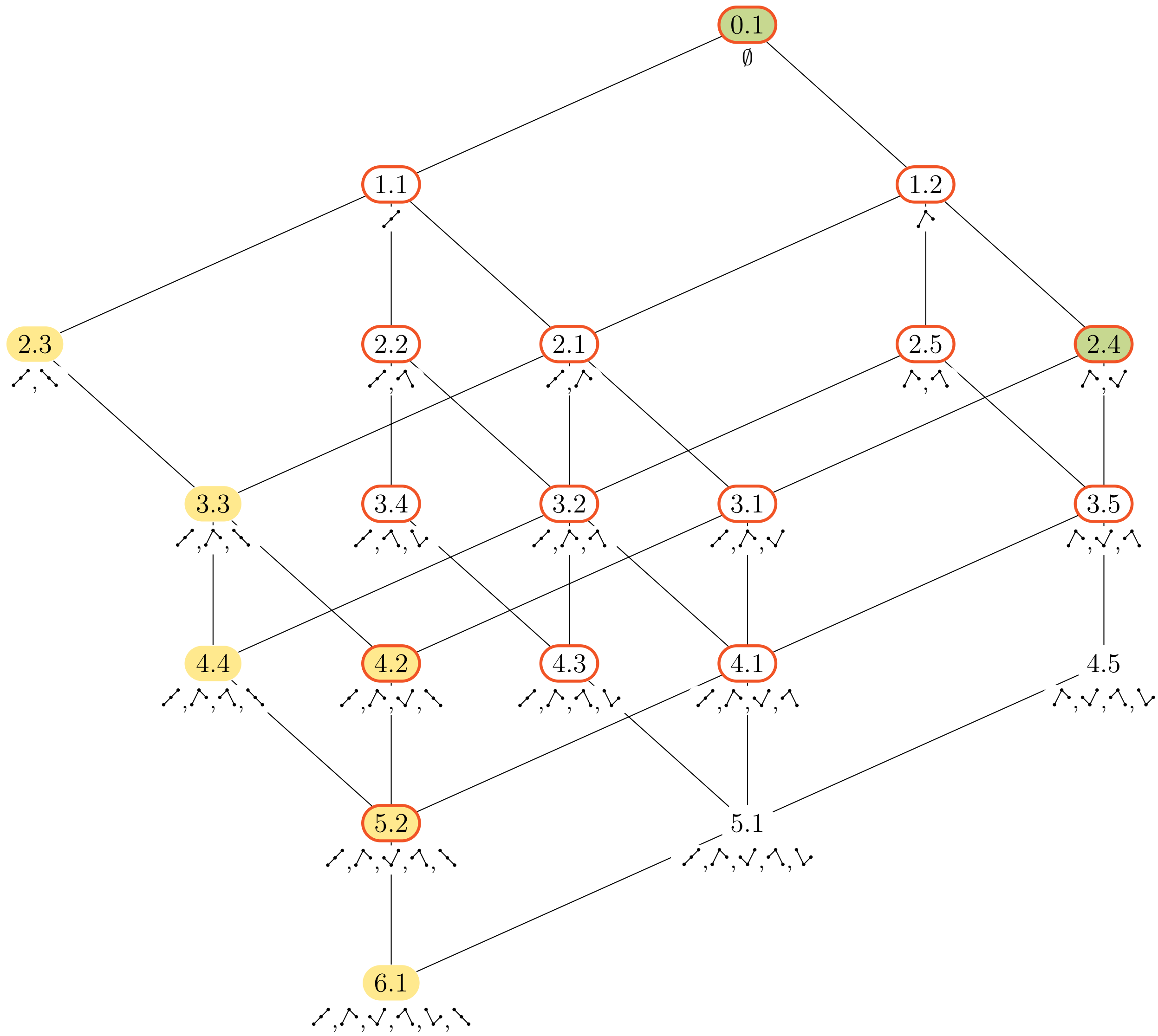


Figure A.3. Subset relation between symmetry classes of sets of patterns. Here the nodes representing classes that generate finite avoidance classes are yellow. The ones representing classes generating Fraïssé classes are green. The nodes circled in red correspond to avoidance classes that have the joint embedding property.

References

[1] Atkinson, M. D.: *Restricted Permutations.* Discrete Mathematics, 195, 1999.

[2] Atkinson, M. D., M. M. Murphy and N. Ruškuc: *Pattern avoidance classes and subpermutations.* Electronic Journal of Combinatorics, 12, 2005.

[3] Böcker, Sebastian, David Bryant, Andreas W. M. Dress and Mike A. Steel: *Algorithmic Aspects of Tree Amalgamation.* Journal of Algorithms, 37, 2000.

[4] Bodirsky, Manuel: *Complexity Classification in Infinite-Domain Constraint Satisfaction.* CoRR, abs/1201.0856v5, 2012.

[5] Bodirsky, Manuel, Peter Jonsson and Trung Van Pham: *The Reducts of the Homogeneous Binary Branching C-relation.* submitted, 2015.

[6] Bóna, Miklós: *Combinatorics of Permutations.* Chapman & Hall/CRC, 2004.

[7] Cameron, Peter J.: *Homogeneous permutations.* Electronic Journal of Combinatorics, 9, 2002.

[8] Cherlin, Gregory L.: *The Classification of Countable Homogeneous Directed Graphs and Countable Homogeneous n-tournaments.* Number 621 in *Memoirs of the American Mathematical Society.* American Mathematical Society, January 1998.

[9] Claesson, Anders and Sergey Kitaev: *Classification of bijections between 321- and 132-avoiding permutations.* In *DMTCS proceedings*, 2008.

[10] Covington, Jacinta: *Homogenizable Relational Structures.* Illinois Journal of Mathematics, 34, 1990.

[11] Erdös, P. and G. Szekeres: *A combinatorial problem in geometry.* In *Compositio Mathematica*, 2, 1935.

[12] Hanika, Tom: *Forbidden permutation patterns and constraint satisfaction problems.* Diploma thesis, Technische Universität Dresden, Institut für Algebra, 2015.

[13] Hodges, Wilfrid: *A shorter model theory.* Cambridge University Press, 1997.

[14] Knott, Gary D.: *A Numbering System for Binary Trees.* Communications of the ACM, vol. 20, February 1977.

[15] Knuth, Donald E.: *The Art of Computer Programming,* Volume 1 / Fundamental Algorithms. Addison-Wesley Publishing Company, 1969.

[16] MacMahon, Percy A.: *Combinatory Analysis.* Cambridge University Press, 1915.

[17] MacPherson, Dougald: *Homogeneous Structures*, 2010. Lecture Notes, Fischbachau MALOA Workshop.

[18] Robinson, Abraham and Jon Barwise: *Completing Theories by Forcing.* Annals of Mathematical Logic, 2(2), 1970.

[19] Rotem, D.: *Stack Sortable Permutations.* Discrete Mathematics, 33, 1981.

[20] Saracino, D.: *Model Companions for* $\aleph_0$*-categorical Theories.* Proceedings of the American Mathematical Society, 39(3), 1973.

[21] Schmerl, James H.: ω*-Categorical Partially Ordered Sets.* In Pouzet, Maurice and Denis Richard (editors): *Orders: Description and Roles*, 99 of *North-Holland Mathematics Studies.* North-Holland, 1984.

[22] Simion, Rodica and Frank W. Schmidt: *Restricted Permutations.* European Journal of Combinatorics, 6, 1985.

[23] Stankova, Zvezdelina E.: *Forbidden subsequences.* Discrete Mathematics, 132, 1994.

[24] Stankova, Zvezdelina E.: *Classification of Forbidden Subsequences of Length 4.* European Journal of Combinatorics, 17, 1996.

[25] Vatter, Vincent: *The Enumeration and Structure of Permutation Classes.* Master's thesis, The State University of New Jersey, 2006.

[26] West, Julian: *Generating Trees and the Catalan and Schröder numbers.* Discrete Mathematics, 146, 1995.